Answer to Quiz 9

By Lei Apr 13, 2011

1. (a). A line can be determined by:__________ (1 pt)(Get all possible answers)
   A. Two points B. Two vectors C. A point on the line and a vector parallel to it
   Ans: A. C.
   For A, two points can determine a line uniquely, since there is only one line that passes through two particular points.
   For B, this is not true. Vectors in plane or space have no fixed positions. As long as two vectors have the same length and directions we regard them to be the same. Thus, no matter what method you use to determine a line (for example, cross product, or something), you actually can't fix the position of the line. B is wrong.
   For C, this is correct. This is only one line that passes through the point and parallel to a vector. We call the vector direction vector.

   (b). Find parametrizations for the line through $A(-1,-3)$,$B(4,1)$ and the line segment with endpoints A, B. (3 pts)
   Ans: We use A and vector $\overrightarrow{AB}$ to get the parametric equations. Notice that I don't want cartesian equations here. $\overrightarrow{AB} =< 5, 4 >$. Then the line can be $x = -1 + 5t, y = -3 + 4t, -\infty < t < +\infty$. For the line segment, t can't be any value, because we only need those points between A and B. The line segment is $x = -1 + 5t, y = -3 + 4t, 0 \leq t \leq 1$.

2. $\overrightarrow{x}(t) = \begin{pmatrix} \sin t \\ t \end{pmatrix}.(-\infty < t < +\infty)$. Sketch the curve and indicate the direction of motion. Can this curve be the graph of a function $y = f(x)$? (3 pts)
   Ans: Since $x = \sin(t)$ and $t = y$, you can get $x = \sin(y)$. Notice that we don't have $y = \sin^{-1}(x)$ because the codomain of $\sin^{-1}$ is $[-\pi/2, \pi/2]$, but our y can achieve values outside this interval.
   Since as t grows, y grows, we can see the direction is upward. This is not the graph of a function, since for one input x, there may be infinitely many y values. Actually, this curve contains the graph of $\sin^{-1}$ as a part, but it has more. However, if we restrict that $-\pi/2 \leq t \leq \pi/2$, then it'll be the graph.

3. $x = t^2, y = t^6 - 2t^4$ $(-\infty < t < +\infty)$. Find the Cartesian equation and sketch roughly (I just need the shape)(2 pts). Indicate the direction of motion.(1 pt)
   Ans: I believe most of you can get $y = x^3 - 2x^2$, however, this is not the complete Cartesian equation, because we only need a part of this function, namely we need to determine the range of x. Since $x = t^2$ where t is real, $x \geq 0$. Then, draw the graph. To see the direction of motion, you can observe the trend of x and y as t grows. Actually, you can see when t moves from $-\infty$ to 0, the particle moves towards the origin. After that the particle moves away following the same path.

Bonus 1: (a). Is parametric equation of one curve unique?(1 pt) Are the two curves the same and why? $x = e^t, y = e^{2t}$ and $x = t, y = t^2$ (1 pt)
Ans: (a). The answer to first question is 'no'. Example: for $y = x^2$, we have $x = t, y = t^2$ and $x = t\sin t, y = (t\sin t)^2$. They have the same range of x and y and the expressions are the same.
The answer to the second quesiton is 'no'. Even if they have the same expression $y = x^2$, the ranges of x are different. This means that the first curve is just the right part of the parabola $y = x^2$ while the second curve is the whole parabola.

(b). Think of an example such that $y = f(x)$ isn't differentiable somewhere, but we can find $x(t), y(t)$ as parametric eqiations that are differentiable everywhere. (Hint: cycloid)(2 pts)
Ans: There are many answers. For example $x(s) = s - \sin s, y(s) = 1 - \cos s$, which is a cycloid and y is a function of x. Notice that $x(t) = s - a\sin(s/R), y = R - a\cos(s/R)$ is cycloid only if $a = R$. These two $x(s)$ and $y(s)$ are differentiable everywhere but $y = y(x)$ is not differentiable at, say, $x = 0$.
Actually, this is interesting. Suppose s is in an open interval. If we require $|r'(s)| \neq 0$ everywhere, then, $y = f(x)$ is differentiable if and only if $x(s), y(s)$ are differentible. Without this, there are exceptions and above we have the exceptions. You can check that $|r'(0)| = 0$ in the cycloid example.

Bonus 2: Suppose the orbit of the earth can be described by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a \neq b)$. What's this curve called and where is the sun? (1 pt) One parametric equation is $x = a\cos(t), y = b\sin(t)$. Is this motion clockwise or counterclockwise?(1 pt). Find one interval of t corresponding to two years on the earth.(1 pt)
Ans: The curve is called ellipse. The sun is at one of the foci. $c^2 = a^2 - b^2$ and the foci are at $(\pm c, 0)$ if $a > b > 0$.
The motion is counterclockwise. Notice that if the equations are $x = a\sin t, y = b\cos t$, then it becomes clockwise.
One interval is $[0, 4\pi)$, because the motion has finished 2 rotations. One rotation is one year. Notice that t is not the number of days. t is also not the angle between the x axis and the line connecting the center and the earth. It's an auxiliary angle. If you draw two circle, one with radius a and the other one with radius b. Draw a vertical line through the earth and intersecting with the outer circle at P. Then t is the angle between OP and the x-axis.