Answer to Quiz 8

By Lei Mar 30, 2011

1. Use the method of undetermined coefficients
   a). $y'' + y = 2x + 3e^x$ (2 pts) b).$y'' + y = \sin x, y(0) = 0, y(\frac{\pi}{2}) = 0$ (3 pts)
   Ans:a).The complementary equation is $y'' + y = 0$ and the corresponding aux. equation is $r^2 + 1 = 0$. We have $r = \pm i$. Then, $y_c = C_1 \cos x + C_2 \sin x$.
   For term $2x$, we can try $Ax + B$. e^x is not a part of the solution to the complementary equation and thus for the second term, we try Ce^x. We try $y_p = Ax + B + Ce^x$.
   Then, we have $y_p'' + y_p = 2x + 3e^x$ and thus $A = 2, B = 0, C = 3/2$.
   The answer is $y = y_c + y_p = C_1 \cos x + C_2 \sin x + 2x + 3e^x/2$

   b). The complementary part is the same as a). We then notice that $\sin x$ is a part of the homogeneous part and thus we try $y_p = D_1 x \cos x + D_2 x \sin x$. Then, we have
   $y_p'' = D_1(-2\sin x - x\cos x) + D_2(2\cos x - x\sin x)$.
   $y_p'' + y_p = -2D_1 \sin x + 2D_2 \cos x = \sin x$. Thus $D_1 = -1/2, D_2 = 0$. We have
   $y_p = -\frac{1}{2} x \cos x$.
   $y = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x$. $y(0) = C_1 + 0 + 0 = 0$. $y(\pi/2) = 0 + C_2 + 0 = 0$.
   We can see that $C_1 = 0, C_2 = 0$.
   Final answer is $y(x) = -\frac{1}{2} x \cos x$

2. 3 pts if variation of parameters and 2 pts otherwise.
   $y'' - y = xe^x$ Just in case you need: y_p has the form $Axe^x + Bx^2e^x$

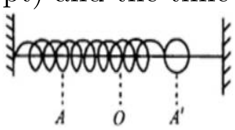
   Ans: For the complementary equation, it's easy to solve $y'' - y = 0$. $r^2 - 1 = 0$ and $r = \pm 1$. $y_c = C_1 e^x + C_2 e^{-x}$
   As I say, if you use the form I give you, you are not using variation of parameters and you can get 2 pts at most. Using this method, you can just plug in and get
   $y_p'' = A(2e^x + xe^x) + B(2e^x + 4xe^x + x^2e^x)$ and then $y_p'' - y_p = (2A + 2B)e^x + 4Bxe^x$.
   Thus $B = 1/4, A = -1/4$. $y = C_1 e^x + C_2 e^{-x} - \frac{1}{4} xe^x + \frac{1}{4} x^2 e^x$
   I want you to use variation of parameters. Assume $y = v_1(x)e^x + v_2(x)e^{-x}$ and impose $v_1'(x)e^x + v_2'(x)e^{-x} = 0$. Then, we can get another equation by plugging this form into the equation $v_1'(x)e^x - v_2'(x)e^{-x} = xe^x$.
   We can get $v_1'(x) = \frac{1}{2}x$ and $v_2'(x) = -\frac{1}{2}xe^{2x}$. Then $v_1(x) = \int \frac{x}{2} dx$ and we pick
   $v_1(x) = x^2/4$. $v_2(x) = \int (-\frac{1}{2}xe^{2x})dx$. Integrating by parts, we get
   $v_2(x) = -\frac{1}{4}xe^{2x} + \frac{1}{8}e^{2x} + C$. We pick $v_2(x) = -\frac{1}{4}xe^{2x} + \frac{1}{8}e^{2x}$. Finally, we have
   $y_p = v_1(x)e^x + v_2(x)e^{-x} = \frac{1}{4}x^2e^x - \frac{1}{4}xe^x + \frac{1}{8}e^x$. Then, general solution is
   $y(x) = C_1 e^x + C_2 e^{-x} + \frac{1}{4}x^2e^x - \frac{1}{4}xe^x + \frac{1}{8}e^x = C_1' e^x + C_2 e^{-x} - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$

3. a). $ay'' + by' + cy = G(x)$. If y_p is a solution, y is any other solution,then $y - y_p = y_c$ is the solution to the complementary equation. (1 pt)
   b). In a, if $G(x) = G_1(x) + G_2(x)$, y_{p1} solves $ay'' + by' + cy = G_1(x)$ and y_{p2} solves $ay'' + by' + cy = G_2(x)$, then y_p can be chosen to be $y_{p1} + y_{p2}$ (1 pt)

Ans: a). Since they are solutions to the equation, we have $ay_p'' + by_p' + cy_p = G(x)$ and $ay'' + by' + cy = G(x)$. Then, we can have
$a(y-y_p)''+b(y-y_p)'+c(y-y_p) = (ay''+by'+cy)-(ay_p''+by_p'+cy_p) = G(x)-G(x) = 0$.
Hence, $y - y_p$ is the solution to the complementary equation and $y = y_c + y_p$. This is the basis for us to solve the inhomogeneous equations.
b). By the definition of solutions, we should have $ay_{p1}'' + by_{p1}' + cy_{p1} = G_1(x)$ and $ay_{p2}'' + by_{p2}' + cy_{p2} = G_2(x)$. Then, we can see that
$a(y_{p1} + y_{p2})'' + b(y_{p1} + y_{p2})' + c(y_{p1} + y_{p2}) = ay_{p1}'' + by_{p1}' + cy_{p1} + ay_{p2}'' + by_{p2}' + cy_{p2} = G_1(x) + G_2(x) = G(x)$. Thus, we can pick y_p to be that form.

Bonus 1: $y''' - 7y' + 6y = x^2$ (2 pts). Hint: For $y''' - 7y' + 6y = 0$, e^x is a solution, so the aux. equation (which exists since coefficients are constants) has a factor $r - 1$.
Ans: Similar to 2nd order ODE with constant coefficients, for 3rd order ODE. Solve the complementary equation first: $y''' - 7y' + 6y = 0$. The corresponding aux. equations can be written as $r^3 - 7r + 6 = 0$. As the hint I give you says, this equation has a factor $r - 1$. Hence we have $r^3 - 7r + 6 = (r - 1) * g(r)$. $g(r)$ can be obtained by long division. The answer is $g(r) = r^2 + r - 6$. Hence, $r^3 - 7r + 6 = (r - 1)(r^2 + r - 6) = (r - 1)(r - 2)(r + 3)$. We have three roots $r = 1, 2, -3$. Then $y_c(x) = C_1e^x + C_2e^{2x} + C_3e^{-3x}$
For the particular solution, it's easy. Notice that the coefficient of y is nonzero, we can try $y_p = Ax^2 + Bx + C$. Then $-7(2Ax + B) + 6(Ax^2 + Bx + C) = x^2$. We have
$A = 1/6, B = 7/18, C = 49/108$. Then
$y(x) = C_1e^x + C_2e^{2x} + C_3e^{-3x} + x^2/6 + 7x/18 + 49/108$

Bonus 2:Simple Harmonic Motion: A mass m is attached on a spring that has a spring constant k. Pull the mass with a displacement $y(0) = C$ from equilibrium position O to A' and then release. Supposing no friction, find the equation the displacement $y(t)$ satisfies (1 pt) and the time needed to reach the midpoint of O and A' for the first time. (2 pts)



Ans: Assume the rightward is positive. Then, the elongation of the spring is exactly the displacement y. By Hook's law, the force of the spring is $-ky$ since right is positive. This is the net force. Newton's law says that $F = ma$, which is exactly $-ky = my''$ and thus $my'' + ky = 0$.
Solve this equation, we have $mr^2 + k = 0$ and $r = \pm i\sqrt{k/m}$. The general solution is $y(t) = C_1 \cos(\sqrt{k/m}t) + C_2 \sin(\sqrt{k/m}t)$. The initial position is $y(0) = C$. The initial velocity is 0 and thus $y'(0) = 0$. We have $C_1 = C, C_2 = 0$. Then $y(t) = C \cos(\sqrt{k/m}t)$. You can also find that the period is $T = 2\pi\sqrt{\frac{m}{k}}$. When it reaches the midpoint, $y(t_1) = C/2$. Thus, $\sqrt{k/m}t_1 = \pi/3$ and we have $t_1 = \frac{\pi}{3}\sqrt{\frac{m}{k}}$.