1. Use the method of undetermined coefficients
   a). \( y'' + y = 2x + 3e^x \) (2 pts)  
   b). \( y'' + y = \sin x, y(0) = 0, y(\frac{\pi}{2}) = 0 \) (3 pts) 
   Ans: a). The complementary equation is \( y'' + y = 0 \) and the corresponding aux. 
equation is \( r^2 + 1 = 0 \). We have \( r = \pm i \). Then, \( y_c = C_1 \cos x + C_2 \sin x \).

   For term \( 2x \), we can try \( Ax + B \). \( e^x \) is not a part of the solution to the complementary equation and thus for the second term, we try \( Ce^x \). We try \( y_p = Ax + B + Ce^x \).

   Then, we have \( y''_p + y_p = 2x + 3e^x \) and thus \( A = 2, B = 0, C = 3/2 \).

   The answer is \( y = y_c + y_p = C_1 \cos x + C_2 \sin x + 2x + 3e^x / 2 \)

b). The complementary part is the same as a). We then notice that \( \sin x \) is a part of the homogeneous part and thus we try \( y_p = D_1 x \cos x + D_2 x \sin x \). Then, we have

   \[
y''_p = D_1(2 \cos x - x \sin x) + D_2(2 \sin x + x \cos x).
   \]

   \[
y''_p + y_p = -2D_1 \sin x + 2D_2 \cos x = \sin x.
   \]

   Thus \( D_1 = -1/2, D_2 = 0 \). We have

   \[
y_p = -\frac{1}{2} x \cos x.
   \]

   \[
y = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x.
   \]

   \[
y(0) = C_1 + 0 + 0 = 0.\]

   \[
y(\pi/2) = 0 + C_2 + 0 = 0.
   \]

   We can see that \( C_1 = 0, C_2 = 0 \).

   Final answer is \( y(x) = -\frac{1}{2} x \cos x \)

2. 3 pts if variation of parameters and 2 pts otherwise.
   \( y'' - y = xe^x \)  
   Just in case you need: \( y_p \) has the form \( Axe^x + Bx^2e^x \)

   Ans: For the complementary equation, it’s easy to solve \( y'' - y = 0 \). \( r^2 - 1 = 0 \) and \( r = \pm 1 \). \( y_c = Ce^x + Ce^{-x} \)

   As I say, if you use the form I give you, you are not using variation of parameters and you can get 2 pts at most. Using this method, you can just plug in and get

   \[
y''_p = A(2e^x + xe^x) + B(2e^x + 4xe^x + x^2e^x)
   \]

   Thus \( B = 1/4, A = -1/4 \). \( y = C_1e^x + C_2e^{-x} - \frac{1}{4} xe^x + \frac{1}{4} x^2e^x \)

   I want you to use variation of parameters. Assume \( y = v_1(x)e^x + v_2(x)e^{-x} \) and impose \( v_1'(x)e^x + v_2'(x)e^{-x} = 0 \). Then, we can get another equation by plugging this form into the equation \( v_1'(x)e^x + v_2'(x)e^{-x} = xe^x \).

   We can get \( v_1'(x) = \frac{1}{2} x \) and \( v_2'(x) = -\frac{1}{2} xe^{2x} \). Then \( v_1(x) = \int \frac{x}{2} dx \) and we pick

   \[
v_1(x) = \frac{x^2}{4}. \quad v_2(x) = \int (-\frac{1}{2} xe^{2x}) dx.
   \]

   Integrating by parts, we get

   \[
v_2(x) = -\frac{1}{4} xe^{2x} + \frac{1}{8} e^{2x} + C.
   \]

   We pick \( v_2(x) = -\frac{1}{4} xe^{2x} + \frac{1}{8} e^{2x} \). Finally, we have

   \[
y_p = v_1(x)e^x + v_2(x)e^{-x} = \frac{1}{4} xe^x - \frac{1}{4} xe^x + \frac{1}{8} e^x.
   \]

   Then, general solution is

   \[
y(x) = C_1e^x + C_2e^{-x} + \frac{1}{4} xe^x - \frac{1}{4} xe^x + \frac{1}{8} e^x = C_1e^x + C_2e^{-x} - \frac{1}{4} xe^x + \frac{1}{4} x^2e^x
   \]

3. a). \( ay'' + by' + cy = G(x) \). If \( y_p \) is a solution, \( y \) is any other solution, then \( y - y_p = y_c \) is the solution to the complementary equation. (1 pt)

   b). In a), if \( G(x) = G_1(x) + G_2(x) \), \( y_{p1} \) solves \( ay'' + by' + cy = G_1(x) \) and \( y_{p2} \) solves \( ay'' + by' + cy = G_2(x) \), then \( y_p \) can be chosen to be \( y_{p1} + y_{p2} \) (1 pt)
Solve this equation, we have \( mr'' + my' + cy = G(x) \). Then, we can have
\[
a(y-y_p)''+b(y-y_p)' + c(y-y_p) = (ay'' + by' + cy) - (ay''_p + by'_p + cy_p) = G(x) - G(x) = 0.
\]
Hence, \( y - y_p \) is the solution to the complementary equation and \( y = y_c + y_p \). This is the basis for us to solve the inhomogeneous equations.

b). By the definition of solutions, we should have \( ay''_p + by'_p + cy_p = G_1(x) \) and \( ay''_p + by'_p + cy_p = G_2(x) \). Then, we can see that
\[
a(y_p1 + y_p2)' + b(y_p1 + y_p2) + c(y_p1 + y_p2) = ay''_p + by'_p + cy_p + ay''_p + by'_p + cy_p = G_1(x) + G_2(x) = G(x).
\]
Thus, we can pick \( y_p \) to be that form.

Bonus 1: \( y'' - 7y' + 6y = x^2 \) (2 pts). Hint: For \( y'' - 7y' + 6y = 0 \), \( e^x \) is a solution, so the aux. equation (which exists since coefficients are constants) has a factor \( r - 1 \).
Ans: Similar to 2nd order ODE with constant coefficients, for 3rd order ODE. Solve the complementary equation first: \( y'' - 7y' + 6y = 0 \). The corresponding aux. equations can be written as \( r^3 - 7r + 6 = 0 \). As the hint I give you says, this equation has a factor \( r - 1 \).
Hence we have \( r^3 - 7r + 6 = (r - 1) \star g(r) \). \( g(r) \) can be obtained by long division. The answer is \( g(r) = r^2 + r - 6 \).
Hence, \( r^3 - 7r + 6 = (r - 1)(r^2 + r - 6) = (r - 1)(r - 2)(r + 3) \).
We have three roots \( r = 1, 2, -3 \). Then \( y_c(x) = C_1e^x + C_2e^{2x} + C_3e^{-3x} \)
For the particular solution, it’s easy. Notice that the coefficient of \( y \) is nonzero, we can try \( y_p = Ax^2 + Bx + C \). Then \( -7(2Ax + B) + 6(Ax^2 + Bx + C) = x^2 \). We have \( A = 1/6, B = 7/18, C = 49/108 \). Then \( y(x) = C_1e^x + C_2e^{2x} + C_3e^{-3x} + x^2/6 + 7x/18 + 49/108 \)

Bonus 2: Simple Harmonic Motion: A mass \( m \) is attached on a spring that has a spring constant \( k \). Pull the mass with a displacement \( y(0) = C \) from equilibrium position \( O \) to \( A' \) and then release. Supposing no friction, find the equation the displacement \( y(t) \) satisfies (1 pt) and the time needed to reach the midpoint of \( O \) and \( A' \) for the first time. (2 pts)

Ans: Assume the rightward is positive. Then, the elongation of the spring is exactly the displacement \( y \). By Hook’s law, the force of the spring is \( -ky \) since right is positive. This is the net force. Newton’s law says that \( F = ma \), which is exactly \( -ky = my'' \) and thus \( my'' + ky = 0 \).
Solve this equation, we have \( mr^2 + k = 0 \) and \( r = \pm i\sqrt{k/m} \). The general solution is \( y(t) = C_1 \cos(\sqrt{k/mt}) + C_2 \sin(\sqrt{k/mt}) \). The initial position is \( y(0) = C \). The initial velocity is 0 and thus \( y'(0) = 0 \).
We have \( C_1 = C, C_2 = 0 \). Then \( y(t) = C \cos(\sqrt{k/mt}) \).
You can also find that the period is \( T = 2\pi\sqrt{m/k} \). When it reaches the midpoint, \( y(t_1) = C/2 \).
Thus, \( \sqrt{k/mt} = \pi/3 \) and we have \( t_1 = \frac{\pi}{3}\sqrt{m/k} \).

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