Answer to Quiz 7

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1. Find the general solutions:
   a) \( y'' - y' - 12y = 0 \) (2 pts)  
   b) \( y'' + 2y' + 4y = 0 \) (3 pts)
   Ans: They are both homogeneous equations with constant coefficients. We can write out the auxiliary equations first and solve.
   a). \( r^2 - r - 12 = 0 \) and we have \( r = 4, -3 \). Two different real roots.
   \( y(x) = C_1 e^{4x} + C_2 e^{-3x} \).
   b). \( r^2 + 2r + 4 = 0 \) and we have \( r = -1 \pm \sqrt{3}i \). Two complex roots.
   \( y(x) = e^{-x}(C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)) \)

2. Find the unique solution to the following: (3 pts)
   \( y'' + 4y' + 4y = 0 \)  
   \( y(0) = 0, y'(0) = 1 \)
   Ans: It’s also a homogeneous equation with constant coefficients.
   \( r^2 + 4r + 4 = 0 \). One repeated root. General solution is
   \( y(x) = C_1 e^{-2x} + C_2 x e^{-2x} \).
   \( y(0) = C_1 = 0 \) and thus \( y(x) = C_2 x e^{-2x} \).
   \( y'(x) = C_2 (-2xe^{-2x} + xe^{-2x}) \). \( y'(0) = C_2 = 1 \). \( y(x) = xe^{-2x} \)

3. a). If \( y_1 \) and \( y_2 \) are solutions to \( y'' - 5y' + 6y = 0 \), how about \( y_1 + y_2 \)? (1 pt)
   b). If \( y_1 \) and \( y_2 \) are solutions to \( y'' - 5y' + 6 = 0 \), how about \( y_1 + y_2 \)? (1 pt)
   Ans: a). It’s homogeneous. We can know \( y_1 + y_2 \) must be the solution. (You can prove like this:
   \( (y_1 + y_2)'' - 5(y_1 + y_2)' + 6(y_1 + y_2) = y_1'' - 5y_1' + 6y_1 + y_2'' - 5y_2' + 6y_2 = 0 + 0 = 0 \). It’s a solution. Actually, following the same process, you can prove \( C_1y_1 + C_2y_2 \) should be solution.)
   b). It’s inhomogeneous. And the sum of coefficients is not 1. It’s not a solution.
   (Generally, you can see \( (C_1y_1 + C_2y_2)'' - 5(C_1y_1 + C_2y_2)' + 6 = C_1(y_1'' - 5y_1') + C_2(y_2'' - 5y_2') + 6 = -6C_1 - 6C_2 + 6 = 0 \).
   Note: You can’t solve it by solving \( r^2 - 5r + 6 = 0 \) because we have 6 instead of 6!)

Bonus 1: If I tell you that two solutions to the equation \( x^2y'' - 5xy' + 9y = 0 \) are of the type \( y_1 = x^r \) and \( y_2 = x^r \ln x \) (here, the two \( r \)'s are the same), which are obviously linearly independent, find \( r \) and write out the general solution. (2 pts)

Ans: You can check that it’s linear and homogeneous. I have told you that the two \( r \)'s are the same. Since the former is simpler, we can plug it in to determine \( r \).
\( x^r \) is the solution means that if we plug it in, we can make the equation hold. \( (x^r)' = rx^{r-1} \)
and \( (x^r)'' = r(r-1)x^{r-2} \). Thus \( x^2(r-1)x^{r-2} - 5rx^{r-1} + 9x^r = 0 \). We have
\( r^2 - 6r + 9 = 0 \) and thus \( r = 3 \). We thus have two linearly independent solutions \( x^3 \) and \( x^3 \ln x \). The general solution should be \( y(x) = C_1 x^3 + C_2 x^3 \ln x \). Notice that
you can **NOT** solve the aux. eqn. \( x^2 r^2 - 5 x r + 9 = 0 \) because it’s not with constant coefficients and it doesn’t have solutions of the form \( e^{rx} \).

**Bonus 2:** Solve \( y'' - 5 y' + 6 = 0 \) (1 pt) and \( 2(y y')' - 10 y y' + 6 y^2 = 0 \) (2 pts)

*Hint:* Note that the first is inhomogeneous and the second is nonlinear! For the first, you can either do substitution \( u = y' \) or use the method you’ll learn soon and for the second, use substitution \( u = y^2 \).

Ans: For the first equation, as the hint says, we can let \( u = y' \) and then we have \( u' - 5u + 6 = 0 \). Somebody got this and couldn’t solve it! This is the first order linear ODE we just learned! The first step, get the integrating factor. \( \mu = e^{\int -5 dx} \). We pick \( \mu = e^{-5x} \).

Then, we have \( (e^{-5x} u)' = -6e^{-5x} \). We have \( u(x) = e^{5x}(6e^{-5x}/5 + C) = 6/5 + Ce^{5x} \).

\( y(x) = \int u(x) dx = 6x/5 + C_1 e^{5x} + C_2 \). Here \( C_1 = C/5 \).

Another method is to use the method for inhomogeneous equation. The complementary equation \( y'' - 5 y' = 0 \). \( y_c = C_1 e^{5x} + C_2 \). Then find a particular solution. \(-6\) is a polynomial. The left hand side has \( y' \) and doesn’t have \( y \). We can try \( y_p = Ax + B \). Then \( A = 6/5, B = 0 \). The final answer is the same.

For the second, \( u = y^2 \) and then \( u' = 2yy' \). Hence the equation becomes \( u'' - 5u' + 6u = 0 \) and we have \( u(x) = C_1 e^{2x} + C_2 e^{3x} \). Then \( y(x) = \pm \sqrt{C_1 e^{2x} + C_2 e^{3x}} \).