Answer to Quiz 7

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1. Find the general solutions:
   a) $y'' - y' - 12y = 0$ (2 pts) b). $y'' + 2y' + 4y = 0$ (3 pts)
   Ans: They are both homogeneous equations with constant coefficients. We can write out the auxiliary equations first and solve.
   a). $r^2 - r - 12 = 0$ and we have $r = 4, -3$. Two different real roots.
   $y(x) = C_1 e^{4x} + C_2 e^{-3x}$.
   b). $r^2 + 2r + 4 = 0$ and we have $r = -1 \pm \sqrt{3}i$. Two complex roots.
   $y(x) = e^{-x}(C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x))$

2. Find the unique solution to the following: (3 pts)
   $y'' + 4y' + 4y = 0 \quad y(0) = 0, y'(0) = 1$
   Ans: It's also a homogeneous equation with constant coefficients.
   $r^2 + 4r + 4 = 0$. $(r+2)^2 = 0$. One repeated root. General solution is
   $y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$.
   $y(0) = C_1 = 0$ and thus $y(x) = C_2 x e^{-2x}$.
   $y'(x) = C_2(e^{-2x} - 2xe^{-2x})$. $y'(0) = C_2 = 1$. $y(x) = xe^{-2x}$

3. a). If y_1 and y_2 are solutions to $y'' - 5y' + 6y = 0$, how about $y_1 + y_2$? (1 pt)
   b). If y_1 and y_2 are solutions to $y'' - 5y' + 6 = 0$, how about $y_1 + y_2$? (1 pt)
   Ans: a). It's homogeneous. We can know $y_1 + y_2$ must be the solution. (You can prove like this:
   $(y_1+y_2)'' - 5(y_1+y_2)' + 6(y_1+y_2) = y_1'' - 5y_1' + 6y_1 + y_2'' - 5y_2' + 6y_2 = 0 + 0 = 0$. It's a solution. Actually, following the same process, you can prove $C_1y_1 + C_2y_2$ should be solution.)
   b). It's inhomogeneous. And the sum of coefficients is not 1. It's not a solution.
   (Generally, you can see $(C_1y_1 + C_2y_2)'' - 5(C_1y_1 + C_2y_2)' + 6 =$
   $C_1(y_1'' - 5y_1') + C_2(y_2'' - 5y_2') + 6 = -6C_1 - 6C_2 + 6 = 6(1 - C_1 - C_2))$
   Note: You can't solve it by solving $r^2 - 5r + 6 = 0$ because we have 6 instead of $6y$!

Bonus 1: If I tell you that two solutions to the equation $x^2y'' - 5xy' + 9y = 0$ are of the type $y_1 = x^r$ and $y_2 = x^r \ln x$ (here, the two r's are the same), which are obviously linearly independent, find r and write out the general solution. (2 pts)

Ans: You can check that it's linear and homogeneous. I have told you that the two $r's$ are the same. Since the former is simpler, we can plug it in to determine r.
x^r is the solution means that if we plug it in, we can make the equation hold. $(x^r)' = rx^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$. Thus $x^2r(r-1)x^{r-2} - 5xrx^{r-1} + 9x^r = 0$. We have $r^2 - 6r + 9 = 0$ and thus $r = 3$. We thus have two linearly independent solutions x^3 and $x^3 \ln x$. The general solution should be $y(x) = C_1x^3 + C_2x^3 \ln x$. Notice that

you can **NOT** solve the aux. eqn. $x^2r^2 - 5xr + 9 = 0$ because it's not with constant coefficients and it doesn't have solutions of the form e^{rx}.

Bonus 2: Solve $y'' - 5y' + 6 = 0$ (1 pt) and $2(yy')' - 10yy' + 6y^2 = 0$ (2 pts)
Hint: Note that the first is inhomogeneous and the second is nonlinear! For the first, you can either do substitution $u = y'$ or use the method you'll learn soon and for the second, use substitution $u = y^2$

Ans: For the first equation, as the hint says, we can let $u = y'$ and then we have $u' - 5u + 6 = 0$. Somebody got this and couldn't solve it! This is the first order linear ODE we just learned! The first step, get the integrating factor. $\mu = e^{\int -5dx}$. We pick $\mu = e^{-5x}$. Then, we have $(e^{-5x}u)' = -6e^{-5x}$. We have $u(x) = e^{5x}(6e^{-5x}/5 + C) = 6/5 + Ce^{5x}$. $y(x) = \int u(x)dx = 6x/5 + C_1e^{5x} + C_2$. Here $C_1 = C/5$.
Another method is to use the method for inhomogeneous equation. The complementary equation $y'' - 5y' = 0$. $y_c = C_1e^{5x} + C_2$. Then find a particular solution. -6 is a polynomial. The left hand side has y' and doesn't have y. We can try $y_p = Ax + B$. Then $A = 6/5, B = 0$. The final answer is the same.

For the second, $u = y^2$ and then $u' = 2yy'$. Hence the equation becomes $u'' - 5u' + 6u = 0$ and we have $u(x) = C_1e^{2x} + C_2e^{3x}$. Then $y(x) = \pm\sqrt{C_1e^{2x} + C_2e^{3x}}$