# Answer to Quiz 6

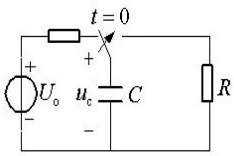
By Lei  Mar 9, 2011

1. Solve the ordinary differential equation $\frac{dy}{dx} = e^{x-y}$ (3 pts)
   Ans: $e^{x-y} = e^x e^{-y}$. It's separable, and thus the solution can be got by: $\int \frac{1}{e^{-y}} dy = \int e^x dx$ and we have $e^y = e^x + C$.
   Note: $y = \ln(e^x + C)$ is neither equal to $\ln e^x + C = x + C$ nor equal to $\ln(e^x)\ln C = x \ln C$!!!

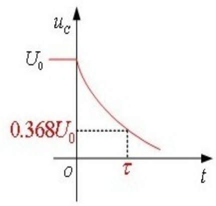
2. $y' + (\tan x)y = \cos^2 x$
   a). Solve it. (3 pts)  b). Check what you get in a) is the solution. (1 pt)
   Ans: This equation is linear, and it's already of the standard form since the coefficient of $y'$ is 1.
   Integrating factor is $\mu(x) = e^{\int \tan x dx}$. Since $\int \tan x dx = \ln|\sec x| + C$, and we can only pick one integrating factor. Let's pick $\mu(x) = \sec x$. Then $\sec x(y' + (\tan x)y) = \sec x \cos^2 x$. Then, we have $(\sec xy)' = \cos x$. Integrate it and we get $\sec xy = \sin x + C$. The solution should be $y(x) = \sin x \cos x + C \cos x$.
   b). We can see $y'(x) = \cos^2 x - \sin^2 x - C \sin x$ and $(\tan x)y = \sin^2 x + C \sin x$. It's clear now that $y' + \tan xy = \cos^2 x$.

3. $xdy + x^4 e^{-x} dx = 3ydx$(Hint: $y' = dy/dx$) (2+1 pts)
   a). If I tell you this is first order linear equation, get the standard form and solve it.
   b). If $y_1(x)$ is the solution satisfying $\lim_{x\to+\infty} y(x)$ exists, find $y_1(x)$ and get the limit.
   Ans:a). By the hint, we can divide by $dx$ first to get $xy' + x^4 e^{-x} = 3y$, which is equivalent to $xy' - 3y + x^4 e^{-x} = 0$.
   It's not of standard form. We can divide by $x$ to get the standard form $y' - \frac{3}{x} y = -x^3 e^{-x}$
   Integrating factor $\mu(x) = e^{\int -3/x dx}$ and we can pick $\mu(x) = e^{-3\ln x} = x^{-3} = 1/x^3$.
   Then $\frac{1}{x^3}(y' - \frac{3}{x}y) = -e^{-x}$ which is $(x^{-3}y)' = -e^{-x}$. We have $y = x^3(e^{-x} + C) = x^3 e^{-x} + Cx^3$
   b). By L'Hopital's rule, $\lim_{x\to+\infty} x^3 e^{-x} = \lim_{x\to+\infty} \frac{x^3}{e^x} = 0$ and if the limit wants to exist, $C$ must to be zero. Hence $y_1(x) = x^3 e^{-x}$ and the limit is 0

(Bonus) In the picture, the electromotive force $U_0 = 1V$, the capacitance $C = 1F$ and the resistance $R = 1\Omega$. At first, the switch was on the left and there was no current. At $t = 0$, we turned the switch to the right.
1). It's known that the charge $q$ on the capacitance and the voltage $u_c$ satisfy $q = Cu_c$. We also know the current $i = \frac{dq}{dt}$. Ohm's Law: the voltage on the resistance is $iR$. Kirchhoff's law: $u_c + iR = 0$. The charge on the capacitance couldn't change immediately and thus the voltage wouldn't change at $t = 0$. Give out the differential equation that $u_c$ satisfies and the initial condition $u_c(0)$. (2 pts)
2). Find the time when the voltage is $e^{-1}V$. (1 pt)




(a)




(b)

Ans: 1). By the equations I gave you: $u_c = -iR = -Ri = -R\frac{dq}{dt} = -R\frac{d(Cu_c)}{dt} = -RC\frac{du_c}{dt}$. Before we turned the switch, the voltage was $U_0$ and I told you that the voltage didn't change and thus $u_c(0) = 1V$.

2). Plugging in $R = 1$, $C = 1$ and $u_c(0) = 1V$, we have $u_c' = -u_c$ and thus $u_c(t) = u_c(0)e^{-t} = e^{-t}$. Thus the time is $\tau = 1(s)$.

Note: Here, $\Omega$, $V$ and $F$ are units, just like $m$(meter) in length and $s$(second) in time. There is no need to put them in equations. Besides, we have $1\Omega \cdot F = 1s$. You can check by the dimensional analysis.