Answer to Quiz 5

By Lei Mar 2, 2011

1. Express $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ as $re^{i\theta}$ where $r>0$ and θ is real. Draw the Argand diagram. (4 pts)
   Ans: Two ways. The first general way is to multiply the conjugate of the denominator on the top and on the bottom at the same time.
   It will be: $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}=\frac{(1+i\sqrt{3})(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})}=\frac{1+2i\sqrt{3}+(i\sqrt{3})^2}{1-(i\sqrt{3})^2}=\frac{-2+i2\sqrt{3}}{4}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$.
   $r=\sqrt{(-1/2)^2+(\sqrt{3}/2)^2}=1$ and draw the picture, you'll find $\theta=2\pi/3$. Answer is $e^{i2\pi/3}$
   The other way is to express the numberater as $r_1e^{i\theta_1}$ and the denominator as $r_2e^{i\theta_2}$. Then, it'll be $\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$. It's not hard to find $1+i\sqrt{3}=2e^{i\pi/3}$ and $1-i\sqrt{3}=2e^{-i\pi/3}$. Final answer is the same.

2. Find the three complex cube roots of -1.(3 pts)
   Ans: Generally, if $z=a+ib=re^{i\theta}$. Suppose the n-the roots are $se^{i\alpha}$. Then $s^ne^{in\alpha}=re^{i\theta}$. Take the absolute values on both sides, we'll have $s^n=r$. Then $e^{in\alpha}=e^{i\theta}$. Hence, $n\alpha=\theta+2k\pi$. We then know that the roots are $r^{1/n}e^{i(\theta+2k\pi)/n}$. We then can see that if $k_1=k_2+2n\pi$, they'll give the same root, so we only need to choose $0\le k\le n-1$.
   For this problem, $r=\sqrt{(-1)^2+0^2}=1$. $\theta=\pi$ by the picture. Then the roots are $e^{i(\pi+2k\pi)/3}$. $k=0$ gives $e^{i\pi/3}=1/2+i\sqrt{3}/2$. $k=1$ gives -1 and $k=2$ gives $1/2-i\sqrt{3}/2$

3. Prove $\sin(2\theta)=2\sin\theta\cos\theta$, $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ by De Moivre's Theorem. (3 pts)
   Ans: $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$, which is obviously true by Euler's identity. This equation is called the De Moivre's Theorem.
   Take $n=2$, and we'll have
   $\cos(2\theta)+i\sin(2\theta)=(\cos\theta+i\sin\theta)^2=(\cos^2\theta-\sin^2\theta)+i2\sin\theta\cos\theta$. Compare the real parts and imaginary parts on both sides and we can get what we want.

Bonus1: True or false? If x is real, $-1\le\cos x\le 1$. If x is complex, $-1\le\cos x\le 1$ (2 pts). Calculate $\cos(i)$ (1 pt)
Ans: If x is real, it's true. You can see this by the graph or by the difinition of function $\cos x$. If x is complex, it's not true. The followed is an example.
$\cos i=(e^{ii}+e^{-ii})/2=(e^{-1}+e)/2>1$. Generally, the cosine function of a complex number is complex, which can't be compared with -1 and 1.

Bonus2: Give an example that e^z can be negative if z is a complex number.(1 pt)
Prove e^z is never zero if z is complex.(Hint: Assume $z=a+bi$) (2 pts)
Ans: Using Euler's identity, we can let $z=i\pi$ to give the first example. We know for real x, e^x is always positive. For complex, this is not true.

$z = a + bi$, $e^z = e^a * e^{ib} = e^a(\cos b + i \sin b)$. $e^a > 0$ and the second part is not zero, because the absolute value is 1. The product thus can't be 0. Actually e^z can achieve any complex number except 0.(0 is called the Picard exception value)