Answer to Quiz 4

By Lei Feb 23, 2011

1. Write out the Taylor series about $a = 0$ for **any two** (no need to explain)(4 pts):
   $\bullet \cos x \quad \bullet \sin x \quad \bullet e^x \quad \bullet \ln(1+x) \quad \bullet \frac{1}{1-x}$
   Ans: Please refer to the book. What I want to mention here is that we can get the expansion for $\ln(1+x)$ by integrating $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$

2. Find the Taylor series of $f(x) = \frac{2+x}{1-x}$ about $a = 0$ (3 pts)
   Ans: We can do like this: $f(x) = \frac{2+x}{1-x} = -1 + \frac{3}{1-x} = -1 + 3\sum_{n=0}^{\infty} x^n = 2 + \sum_{n=1}^{\infty} 3x^n$
   Besides, you can calculate the derivatives. $f'(x) = \frac{3}{(1-x)^2}$ and thus generally, $f^{(n)}(x) = \frac{3*n!}{(1-x)^n}$.

3. Estimate the error if we use $p(x) = 1 + x/2$ to approximate $f(x) = \sqrt{1+x}$ when $|x| < 0.01$ (3 pts)
   Ans: You can check that $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$. Hnece, $f(0) = 1, f'(0) = 1/2, f''(0) = -1/4$. We can see clearly that $p(x)$ is $T_1^0 f(x)$. Thus, the error is $R_1(x) = f(x) - T_1 f(x) = \frac{f^{(2)}(c)}{2!}x^2$, where c is between 0 and x.
   Notice that $|x| < 0.01$ (Not the error is less than 0.01!), and x can be negative, we conclude that $-0.01 < c < 0.01$. Then $|f''(c)| = |\frac{1}{(1+c)^{3/2}}| < \frac{1}{0.99^{3/2}}$. We have $|error| = |R_1(x)| < \frac{1}{8} * \frac{1}{0.99^{3/2}} * 0.01^2$

Bonus: Find the Maclaurin series for $f(x) = \int_0^x t^2 e^{2t^2} dt$ (2 pts)
Ans: $t^2 e^{2t^2} = t^2 \sum_{n=0}^{\infty} \frac{(2t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n t^{2n+2}}{n!}$.
Then $f(x) = \sum_{n=0}^{\infty} \int_0^x \frac{2^n t^{2n+2}}{n!} dt = \sum_{n=0}^{\infty} \frac{2^n}{(2n+3)n!} x^{2n+3}$