Answer to Quiz 4

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1. Write out the Taylor series about \( a = 0 \) for \textbf{any two} (no need to explain)\((4 \text{ pts})\):

\[ \cos x \quad \sin x \quad e^x \quad \ln(1 + x) \quad \frac{1}{1-x} \]

Ans: Please refer to the book. What I want to mention here is that we can get the expansion for \( \ln(1 + x) \) by integrating \( \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \)

2. Find the Taylor series of \( f(x) = \frac{2+x}{1-x} \) about \( a = 0 \) \((3 \text{ pts})\)

Ans: We can do like this:

\[ f(x) = \frac{2+x}{1-x} = -1 + \frac{3}{1-x} = -1 + 3 \sum_{n=0}^{\infty} x^n = 2 + \sum_{n=1}^{\infty} 3x^n \]

Besides, you can calculate the derivatives. \( f'(x) = \frac{3}{(1-x)^2} \) and thus generally, \( f^{(n)}(x) = \frac{3^n n!}{(1-x)^n} \).

3. Estimate the error if we use \( p(x) = 1 + x/2 \) to approximate \( f(x) = \sqrt{1+x} \) when \(|x| < 0.01 \) \((3 \text{ pts})\)

Ans: You can check that \( f'(x) = \frac{1}{2}(1+x)^{-1/2}, f''(x) = -\frac{1}{4}(1+x)^{-3/2} \). Hence, \( f(0) = 1, f'(0) = 1/2, f''(0) = -1/4 \). We can see clearly that \( p(x) \) is \( T_1^0 f(x) \). Thus, the error is \( R_1(x) = f(x) - T_1 f(x) = \frac{f^{(2)}(c)}{2!} x^2 \), where \( c \) is between 0 and \( x \).

Notice that \(|x| < 0.01 \) (Not the error is less than 0.01!), and \( x \) can be negative, we conclude that \(-0.01 < c < 0.01\). Then \(|f''(c)| = \left| \frac{1}{(1+c)^{3/2}} \right| < \frac{1}{0.99972} \). We have \(|\text{error}| = |R_1(x)| < \frac{1}{8} \times \frac{1}{0.99972} \times 0.01^2 \)

Bonus: Find the Maclaurin series for \( f(x) = \int_0^x t^2 e^{2t} dt \) \((2 \text{ pts})\)

Ans: \( t^2 e^{2t} = t^2 \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n t^{2n+2}}{n!} \).

Then \( f(x) = \sum_{n=0}^{\infty} \int_0^x \frac{2^n t^{2n+2}}{n!} dt = \sum_{n=0}^{\infty} \frac{2^n}{(2n+3)n!} x^{2n+3} \)