Keys to Quiz3
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1. Calculate the sums of the following two:

1). \( \sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} \) (3 pts)

Ans: You can check that \( a_{n+1}/a_n = -\frac{1}{4} \), which is a constant. Thus this series is a geometric series. The multiplier is exactly \( r = -\frac{1}{4} \). We also have \( |r| < 1 \). The first term is 5. Then the sum should be \( S = \frac{\text{first}}{1-r} = \frac{5}{1-(-\frac{1}{4})} = 4 \)

Attention: Some may notice that this is also a convergent alternating series. Yes, it is! However, the alternating series test only tells us that it converges, but it can’t tell us what the sum is.

2). \( \sum_{n=0}^{\infty} \left( \sqrt[n]{n} - \frac{n}{\sqrt[n]{n} + 1} \right) \) (2 pts)

Ans: This series is obviously telescoping series. The n-th partial sum is:

\[ S_n = (1 - \sqrt{2}) + (\sqrt{2} - \sqrt{3}) + \ldots + \left( \sqrt{n} - \frac{n}{\sqrt[n]{n} + 1} \right) = 1 - \frac{n}{\sqrt[n]{n} + 1} . \]

Taking the limit and using \( \lim_{n \to \infty} n^{1/n} = 1 \), we have \( S = 1 - 1 = 0 \)

2. Determine whether the serieses converge or diverge and give your reasons:

1) \( \sum_{n=5}^{\infty} \cos\left(\frac{1}{n}\right) \) (2 pts)

2) \( \sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)} \) (3 pts)

Ans: Given a series, the first thing is to use n-th term test for divergence.

1). \( \lim_{n \to \infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1 \), which is not zero. The series diverges.

2). \( \lim_{n \to \infty} a_n = 0 \). We can’t have any conclusions. We must use other methods.

Since \( f(x) = \frac{1}{x(1+\ln^2 x)} \) on \( x > 1 \) is positive decreasing. We can use Integral test. The corresponding integral is:

\[ \int_{1}^{\infty} \frac{1}{x(1+\ln^2 x)} \, dx \]

Use subtitution \( u = \ln x \), we have \( \int_{0}^{\infty} \frac{1}{1+u^2} \, du = \pi/2 \). This improper integral converges and so does the series.
Bonus 1: Choose one for each of the following: A. \( p \geq 1 \)  B. \( p > 1 \)  C. \( p \leq 1 \)  D. \( p < 1 \).

1). When does \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converge? (1 pt)  
2). When does \( \sum_{n=2}^{\infty} \frac{1}{n^p \ln^2 n} \) converge? (1 pt)

Ans: 1) Use the integral test, we can determine easily that the answer is B.

2). If \( p > 1 \), we know that \( \frac{1}{n^p \ln^2 n} < \frac{1}{n^p} \) for \( n > 3 \) and by the direct comparison test, it converges.

If \( p = 1 \), use the integral test, we know it also converges.

If \( p < 1 \), we must use other methods. Notice that as for the speeds of going to infinity, we have \((\ln n)^q \ll n^r \ll a^n (a > 1) \ll n! \ll n^n\). \((q > 0, r > 0)\). I mean we have \( \lim_{n \to \infty} \frac{(\ln n)^q}{n^r} = 0 \) for any \( q > 0, r > 0 \). We then rewrite \( \frac{1}{n^p \ln^2 n} = \frac{1}{n^{(p+1)/2} \ln^3 n} \). Since \( (p+1)/2 < 1 \) and \((1-p)/2 > 0\), we have \( \lim_{n \to \infty} \frac{\ln^2 n}{n^{(1-p)/2}} = 0 \) and thus for large \( n \), we have \( \frac{n^{(1-p)/2}}{\ln^2 n} > 1 \). Then, for these big enough \( n \), we have \( \frac{1}{n^p \ln^2 n} > \frac{1}{n^{(p+1)/2}} \). The sum of the latter is part of the \( p \)-series with the index smaller than 1, and hence it diverges. By comparison test, the former also diverges. The answer is A.

Bonus 2: If I tell you that \( u_n = \frac{n+5}{n+3} \) is positive and decreasing, does the AST (Alternating Series Test) apply for \( \sum_{n=1}^{\infty} (-1)^n \frac{n+5}{n+3} \)? Why? Does this series converge? (2 pts)

Ans: The AST doesn’t apply here, because the positive part doesn’t go to zero. This series doesn’t converge because the \( n \)-th term doesn’t go to zero.