Keys to Quiz3

By Lei February 9, 2011

1. Calculate the sums of the following two:

   1). $\sum\limits_{n=0}^{\infty}(-1)^n \frac{5}{4^n}$ (3 pts)

   Ans: You can check that $a_{n+1}/a_n = -\frac{1}{4}$, which is a constant. Thus this series is a geometric series. The multiplier is exactly $r = -\frac{1}{4}$. We also have $|r| < 1$. The first term is 5. Then the sum should be $S = \frac{first}{1-r} = \frac{5}{1-(-1/4)} = 4$

   Attention: Some may notice that this is also a convergent alternating series. Yes, it is! However, the alternating series test only tells us that it converges, but it can't tell us what the sum is.

   2). $\sum\limits_{n=1}^{\infty}(\sqrt[n]{n} - \sqrt[n+1]{n+1})$ (2 pts)

   Ans: This series is obviously telescoping series. The n-th partial sum is:
   $S_n = (1 - \sqrt{2}) + (\sqrt{2} - \sqrt[3]{3}) + \ldots + (\sqrt[n]{n} - \sqrt[n+1]{n+1}) = 1 - \sqrt[n+1]{n+1}$.
   Taking the limit and using $\lim\limits_{n\to\infty} n^{1/n} = 1$, we have $S = 1 - 1 = 0$

2. Determine whether the serieses converge or diverge and give your reasons:

   1) $\sum\limits_{n=5}^{\infty}\cos(\frac{1}{n})$ (2 pts) 2) $\sum\limits_{n=1}^{\infty}\frac{1}{n(1+\ln^2 n)}$ (3 pts)

   Ans: Given a series, the first thing is to use n-th term test for divergence.
   1). $\lim\limits_{n\to\infty}\cos(1/n) = \cos 0 = 1$, which is not zero. The series diverges.
   2). $\lim\limits_{n\to\infty} a_n = 0$. We can't have any conclusions. We must use other methods.
   Since $f(x) = \frac{1}{x(1+\ln^2 x)}$ on $x > 1$ is positive decreasing. We can use Integral test. The corresponding integral is:
   $\int_1^{\infty}\frac{1}{x(1+\ln^2 x)}dx$. Use substituion $u = \ln x$, we have $\int_0^{\infty}\frac{1}{1+u^2}du = \pi/2$. This improper integral converges and so does the series.

Bonus 1: Choose one for each of the following: A. $p \geq 1$ B. $p > 1$ C. $p \leq 1$ D. $p < 1$.
1). When does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? (1 pt) 2). When does $\sum_{n=2}^{\infty} \frac{1}{n^p \ln^2 n}$ converge? (1 pt)
Ans:1) Use the integral test, we can determine easily that the anwser is B.
2). If $p > 1$, we know that $\frac{1}{n^p \ln^2 n} < \frac{1}{n^p}$ for $n > 3$ and by the direct comparison test, it converges.
If $p = 1$, use the integral test, we know it also converges.
If $p < 1$, we must use other methods. Notice that as for the speeds of going to infinity, we have $(\ln n)^q << n^r << a^n (a > 1) << n! << n^n$. $(q > 0, r > 0)$. I mean we have $\lim_{n\to\infty} \frac{(\ln n)^q}{n^r} = 0$ for any $q > 0, r > 0$. We then rewrite $\frac{1}{n^p \ln^2 n} = \frac{1}{n^{(p+1)/2}} \frac{n^{(1-p)/2}}{\ln^2 n}$. Since $(p+1)/2 < 1$ and $(1-p)/2 > 0$, we have $\lim_{n\to\infty} \frac{\ln^2 n}{n^{(1-p)/2}} = 0$ and thus for large n, we have $\frac{n^{(1-p)/2}}{\ln^2 n} > 1$. Then, for these big enough n, we have $\frac{1}{n^p \ln^2 n} > \frac{1}{n^{(p+1)/2}}$. The sum of the latter is part of the p-series with the index smaller than 1, and hence it diverges. By comparison test, the former also diverges. The answer is A.

Bonus 2: If I tell you that $u_n = \frac{n+5}{n+3}$ is positive and decreasing, does the AST(Alternating Series Test) apply for $\sum_{n=1}^{\infty} (-1)^n \frac{n+5}{n+3}$? Why? Does this series converge? (2 pts)
Ans: The AST doesn't apply here, because the positive part doesn't go to zero. This series doesn't converge because the n-th term doesn't go to zero.