Keys to Quiz2

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1. \( \int \frac{x^2}{x^2+1} \, dx \) (5 pts)
   Ans: Since \( \text{deg}(x^4) = 4 \), \( \text{deg}(x^2 - 1) = 2 \) and \( 4 > 2 \), we must use long division to reduce this improper fraction to a polynomial plus a proper fraction. Only the proper fraction has a partial fraction expression. Using long division, we have \( \frac{x^2}{x^2+1} = x^2 + 1 + \frac{1}{x^2 + 1} \). Then let \( \frac{1}{x^2 + 1} = \frac{A}{x - 1} + \frac{B}{x + 1} \) and we can then determine \( A = 1/2 \), \( B = -1/2 \). The integral then becomes:
   \[ \int (x^2 + 1 + \frac{1}{x - 1} - \frac{1}{x + 1}) \, dx = \frac{1}{3} x^3 + x + \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C \]

2. \( \int_{\sqrt{3}}^{+\infty} \frac{x^2 - x + 1}{(x-1)^2 (x^2 + 1)} \, dx \) (2 pts)
   Ans: This problem is quite hard, and thus I only let it have 2 points.
   We can check that the degree of the numerator is 2 and the degree of the denominator is 4, and thus this fraction is already proper. We then have the following partial fraction expression:
   \( \frac{x^2 - x + 1}{(x-1)^2 (x^2 + 1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x + 1} + \frac{D}{x^2 + 1} \). Multiplying the denominator, we have:
   \( x^2 - x + 1 = A(x^2 + 1) + B(x - 1)(x^2 + 1) + (Cx + D)(x - 1)^2 \).
   Choose a good \( x \), and we let \( x = 1: 1 = 2A \). We have \( A = 1/2 \).
   Then compare the coefficients or let \( x \) to be 0, -1, 2 or something else. Here, I’ll compare the coefficients:
   \( 1 : 1 = A - B + D \). \( x^3 : 0 = B + C \). \( x^2 : 1 = A - B - 2C + D \). Since we know \( A = 1/2 \) and we have three equations together with three unknowns. We can then solve it now:
   \( A = D = 1/2, B = C = 0 \). Then we have:
   \[ \int_{\sqrt{3}}^{+\infty} \frac{x^2 - x + 1}{(x-1)^2 (x^2 + 1)} \, dx = \int_{\sqrt{3}}^{+\infty} \left( \frac{1/2}{(x-1)^2} + \frac{1/2}{x^2 + 1} \right) \, dx = \lim_{b \to \infty} \int_{\sqrt{3}}^{b} \left( \frac{1/2}{(x-1)^2} + \frac{1/2}{x^2 + 1} \right) \, dx = \lim_{b \to \infty} \left( \frac{1}{2} \frac{1}{b - 1} + \frac{1}{2} \tan^{-1}(x) \right)_{\sqrt{3}}^{b} = \frac{1}{2} \frac{1}{\sqrt{3} - 1} + \frac{1}{2} \left( \frac{\pi}{2} \right) = \frac{\pi}{3} + \frac{\pi}{12} \]

3. Determine whether the improper integral converges or diverges: \( \int_{1}^{+\infty} \frac{dx}{x^2 + \tan x} \) (3 pts)
   Ans: We can use comparison test to do. One is limit comparison test and one is direct comparison test. Here I’ll use direct comparison test.
   Notice that \(-1 \leq \sin x \leq 1\). We have \( \frac{1}{x^2 + \tan x} \leq \frac{1}{x^2 - 1} \leq \frac{1}{x^2} \) since \( x \geq 1 \). Attention: we don’t have \( \frac{1}{x^2 + \sin x} \leq \frac{1}{x^2} \! \)!
   Since \( \int_{1}^{+\infty} \frac{dx}{x^2} \) converges, we know the original integral converges by direct comparison test.
Bonus 1: For which \( \alpha \)'s do the integrals converge: \( \int_1^\infty \frac{1}{x^\alpha} \, dx \), \( \int_0^1 \frac{1}{x^\alpha} \, dx \), \( \int_0^\infty \frac{1}{x^\alpha} \, dx \)? (3 pts)

Ans:
For the first integral, it’s of the first type. It is well defined everywhere. The only problem is the range is infinity. By definition, we have:
\[
\int_1^\infty \frac{1}{x^\alpha} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^\alpha} \, dx.
\]
If \( \alpha = 1 \), then it is \( \lim \ln b \) which diverges. If it is not 1,
\[
\lim_{b \to \infty} \frac{1}{1-\alpha} b^{1-\alpha} - \frac{1}{1-\alpha}.
\]
If we want the first term to have a finite limit, we must require
\[
1-\alpha < 0, \text{ which means } \alpha > 1.
\]
We finally need \( \alpha > 1 \).

For the second integral, if \( \alpha \leq 0 \), then it is normal definite integral and it’s well defined. If \( \alpha > 0 \), the function blows up around 0. By definition, \( \lim \int_0^a x^{-\alpha} \, dx \). If \( \alpha = 1 \), it diverges since \( \ln a \) goes to negative infinity as a goes to 0 from right. If it’s not 1, then it becomes \( \lim_{a \to 0^+} \left( \frac{1}{1-\alpha} - \frac{1}{1-\alpha} a^{1-\alpha} \right) \). If we want the second term to be finite, we must require \( 1-\alpha > 0 \).

We have \( 0 < \alpha < 1 \). Together with the normal definite integral case, we have \( \alpha < 1 \).

For the third, it’s a combination of the first two cases. The integral converges if and only if both of them converge. However, for any \( \alpha \), this can’t be true.

Bonus 2: Converges or diverges \( \int_0^{+\infty} x^5 e^{-x^2} \, dx \)? (2 pts)

Ans: We have \( \lim x^n e^{-x^2} = 0 \) for any \( n \). Then if we choose \( n = 7 \) and for large enough \( x \), we have \( x^5 e^{-x^2} < 1/x^2 \). By direct comparison test, this integral converges.

I didn’t require you to get the integral. However, the result can be calculated accurately. Just do substituition \( u = x^2 \) and then apply integral by parts. Many of you did with this method.