Keys to Quiz2

By Lei February 2, 2011

1. $\int \frac{x^4}{x^2-1}\mathrm{d}x$ (5 pts)
   Ans: Since $deg(x^4) = 4$, $deg(x^2-1) = 2$ and $4 > 2$, we must use long division to reduce this improper fraction to a polynomial plus a proper fraction. Only the proper fraction has a partial fraction expression. Using long division, we have $\frac{x^4}{x^2-1} = x^2+1+\frac{1}{x^2-1}$. Then let $\frac{1}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1}$ and we can then determine $A = 1/2, B = -1/2$. The integral then becomes:
   $\int (x^2+1+\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{1}{x+1})dx = \frac{1}{3}x^3+x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$

2. $\int_{\sqrt{3}}^{+\infty} \frac{x^2-x+1}{(x-1)^2(x^2+1)}\mathrm{d}x$ (2 pts)
   Ans: This problem is quite hard, and thus I only let it have 2 points.
   We can check that the degree of the numerater is 2 and the degree of the denominator is 4, and thus this fraction is already proper. We then have the following partial fraction expression:
   $\frac{x^2-x+1}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}$. Multiplying the denominator, we have:
   $x^2-x+1 = A(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)^2$.
   Choose a good x, and we let $x = 1$: $1 = 2A$. We have $A = 1/2$.
   Then compare the coefficients or let x to be $0, -1, 2$ or something else. Here, I'll compare the coefficients:
   $1 : 1 = A-B+D$. $x^3 : 0 = B+C$. $x^2 : 1 = A-B-2C+D$. Since we know $A = 1/2$ and we have three equations together with three unknowns. We can then solve it now: $A = D = 1/2, B = C = 0$. Then we have:
   $\int_{\sqrt{3}}^{+\infty} \frac{x^2-x+1}{(x-1)^2(x^2+1)}\mathrm{d}x = \int_{\sqrt{3}}^{+\infty}(\frac{1/2}{(x-1)^2}+\frac{1/2}{x^2+1})dx = \lim_{b\to\infty}\int_{\sqrt{3}}^{b}(\frac{1/2}{(x-1)^2}+\frac{1/2}{x^2+1})dx =$
   $\lim_{b\to\infty}(-\frac{1}{2}\frac{1}{x-1}+\frac{1}{2}\tan^{-1}(x))|_{\sqrt{3}}^{b} = \frac{1}{2}\frac{1}{\sqrt{3}-1}+\frac{1}{2}(\lim_{b\to\infty}\tan^{-1}(b)-\tan^{-1}(\sqrt{3})) = \frac{\sqrt{3}+1}{4}+\frac{\pi}{12}$

3. Determine whether the improper integral converges or diverges: $\int_1^{\infty}\frac{dx}{2x^3+\sin x}$ (3 pts)
   Ans: We can use comparison test to do. One is limit comparison test and one is direct comparison test. Here I'll use direct comparison test.
   Notice that $-1 \leq \sin x \leq 1$. We have $\frac{1}{2x^3+\sin x} \leq \frac{1}{2x^3-1} \leq \frac{1}{x^3}$ since $x \geq 1$. Attention: we don't have $\frac{1}{2x^3+\sin x} \leq \frac{1}{2x^3}$!
   Since $\int_1^{\infty}\frac{dx}{x^3}$ converges, we know the original integral converges by direct comparison test.

Bonus 1: For which α's do the intergrals converge: $\int_1^\infty \frac{1}{x^\alpha}dx$, $\int_0^1 \frac{1}{x^\alpha}dx$, $\int_0^\infty \frac{1}{x^\alpha}dx$? (3 pts)
Ans:
For the first integral, it's of the first type. It is well defined everywhere. The only problem is the range is infinity. By definition, we have:
$\int_1^\infty \frac{1}{x^\alpha}dx = \lim\limits_{b\to\infty} \int_1^b \frac{1}{x^\alpha}dx$. If $\alpha = 1$, then it is $\lim\limits_{b\to\infty} \ln b$ which diverges. If it is not 1, $\lim\limits_{b\to\infty} \frac{1}{1-\alpha}b^{1-\alpha} - \frac{1}{1-\alpha}$. If we want the first term to have a finite limit, we must require $1-\alpha < 0$, which means $\alpha > 1$. We finally need $\alpha > 1$.

For the second integral, if $\alpha \leq 0$, then it is normal definite integral and it's well defined. If $\alpha > 0$, the function blows up around 0. By definition, $\lim\limits_{a\to 0^+} \int_a^1 x^{-\alpha}dx$. If $\alpha = 1$, it diverges since $\ln a$ goes to negative infinity as a goes to 0 from right. If it's not 1, then it becomes $\lim\limits_{a\to 0^+} (\frac{1}{1-\alpha} - \frac{1}{1-\alpha}a^{1-\alpha})$. If we want the second term to be finite, we must require $1-\alpha > 0$. We have $0 < \alpha < 1$. Together with the normal definite integral case, we have $\alpha < 1$

For the third, it's a combination of the first two cases. The integral converges if and only if both of them converge. However, for any α, this can't be true.

Bonus 2: Converges or diverges $\int_0^{+\infty} x^5 e^{-x^2}dx$? (2 pts)
Ans: We have $\lim\limits_{x\to\infty} x^n e^{-x^2} = 0$ for any n. Then if we choose $n = 7$ and for large enough x, we have $x^5 e^{-x^2} < 1/x^2$. By direct comparison test, this integral converges.
I didn't require you to get the integral. However, the result can be calculated accurately. Just do substituion $u = x^2$ and then apply integral by parts. Many of you did with this method.