

Answer to Quiz 11

By Lei Apr. 27, 2011

1. $r^2 = \cos \theta$. Identify symmetries. Sketch curve and get the slope at $(r, \pi/3)$ $r > 0$.(6')
Ans: Notice $r > 0$ in the problem is to indicate the location of the point and it's not used to give restrictions on the curve. But don't worry, I didn't subtract point.

Suppose (r, θ) is on the curve, and then $r^2 = \cos \theta$ should be satisfied. We can find the coordinate of the point which is symmetric to it about x -axis. The coordinates are $(r, -\theta + 2k\pi)$ and $(-r, \pi - \theta + 2k\pi)$. If one of these expression also satisfies the equation, then the curve is symmetric about x -axis. We can see that $\cos(-\theta) = \cos(\theta)$, so $r^2 = \cos(-\theta)$ and thus $(r, -\theta)$ is on the curve. x is OK.

How about y ? The coordinates are $(r, \pi - \theta + 2k\pi)$ and $(-r, -\theta + 2k\pi)$. We can see even though $(r, \pi - \theta)$ doesn't work. $(-r, -\theta)$ will work. y is OK.

Above two have implied the symmetry about the origin. However, we can verify directly. $(-r, \theta + 2k\pi)$ and $(r, \pi + \theta + 2k\pi)$. We can see $(-r, \theta)$ works.

To draw the picture, you can pick many enough (r_i, θ_i) pairs to plot and then connect. Also you can analyze the trend of r with θ . I won't draw in computer. You can use Matlab, MFC, or Java etc to draw. I think Matlab is the easiest. Using Java, you may use *swing*.

Let's go to slope. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$.

When $\theta = \pi/3$, $r = \frac{1}{\sqrt{2}}$ since $r > 0$. We can see that we only need $\frac{dr}{d\theta}|_{\theta=\pi/3}$ now. To do this, we have two methods. One is to use $r = \sqrt{\cos \theta}$ and $dr/d\theta = \frac{-\sin \theta}{2\sqrt{\cos \theta}}$ and plug in. $r' = -\frac{\sqrt{3}}{2\sqrt{2}}$. I like another method. $r^2 = \cos \theta$. Take derivative with respect to θ on both sides. $2rr' = -\sin \theta$. We have $r' = -\sin \theta/(2r)$. Plug in $r'(\pi/3) = -\frac{\sqrt{3}}{2\sqrt{2}}$. Then, after some calculation $\frac{dy}{dx} = \frac{1}{3\sqrt{3}}$.

2. (a). The 'love' curve $r = 1 + \cos \theta$ is a cardioid. Get its length and the area inside.(4')
Answer: Draw the rough picture first. I won't draw that here. Then you may want to pick a start point and an end point. In discussion, I picked the start and end points as the origin. At the origin, it's not clear what the angles should be by eye. You must do some calculation to determine the direction of the rays. The method is to notice that $r = 0$ at the origin and then $\cos \theta = -1$. $-\pi, \pi$ should be the limits. If we go from the right most point, it's easy to see the initial angle should be zero, because the ray starts at the origin and through the point is horizontal. Similarly, the end angle is 2π . I'll use 0 and 2π as example now.

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta = \int_0^{2\pi} \frac{1}{2} (1 + 2\cos \theta + \cos^2 \theta) d\theta$$

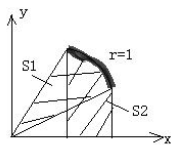
$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$. Using this, you can determine $A = \frac{3}{2}\pi$.

$$\text{For the length, } L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta$$

$= \int_0^{2\pi} \sqrt{2 + 2 \cos \theta} d\theta$. Using $1 + \cos \theta = 2 \cos^2(\theta/2)$, we have

$$L = \int_0^{2\pi} 2 |\cos(\theta/2)| d\theta = \int_0^\pi 2 \cos(\theta/2) d\theta - \int_\pi^{2\pi} 2 \cos(\theta/2) d\theta = 8$$

(b). In the figure below, $\pi/6 \leq \theta \leq \pi/3$. **Set up** the integrals for S_1 and S_2 .(2')



Ans: S_1 is the area bound by the rays from the origin and the curve. Thus we use $\int \frac{1}{2} r^2 d\theta$. The answer is $S_1 = \int_{\pi/6}^{\pi/3} \frac{1}{2} 1^2 d\theta$. For S_2 , it's bounded by the vertical lines.

Thus we use $\int y(\theta) x'(\theta) d\theta$. $y = r \sin \theta = \sin \theta$ and $x = \cos \theta$. The answer is

$$S_2 = \int_{\pi/6}^{\pi/3} (-\sin^2 \theta) d\theta$$

3. (a). $x^2 + y^2 = 1$ is a circle in plane and a _____ in space. (1')
 (b). Find the radius and center of $x^2 + y^2 + z^2 + 4x - 4z = 0$.(2') Sketch the tangent line on the top and parallel to x -axis. (No need to draw the sphere)(1')

Ans: (a). The answer should be cylinder. It's not solid. Actually, it's just the surface of an infinitely long solid cylinder. You can imagine moving the circle up and down since it has nothing to do with z .

(b). Completing the square, you'll have $(x + 2)^2 + y^2 + (z - 2)^2 = 8 = r^2$. Thus the center is $(-2, 0, 2)$ and $r = 2\sqrt{2}$. The tangent line should be $2\sqrt{2}$ above the center. Thus it's in xz -plane and it hits z -axis at $(0, 0, 2 + 2\sqrt{2})$.