Answer to Quiz 10

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1. Find the slope of the curve $x = t^3 + t, y + 2t^3 = 2x + t^2$ at $t = 1$. $(2')$ Get the tangent line there. $(1')$
   Ans: The slope is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. One way is to get the expression of $y(t)$:
   $y = 2x + t^2 - 2t^3 = 2(t^3 + t) + t^2 - 2t^3 = 2t + t^2$. Thus $dy/dt = 2 + 2t$ and $dx/dt = 3t^2 + 1$. Another method is to take the derivative directly:
   $y'(t) + 6t^2 = 2x'(t) + 2t$ and $x'(t) = 3t^2 + 1$. No matter what way you use, plug in $t = 1$, you'll get $x'(1) = 4$ and $y'(1) = 4$. At $t = 1$, $x = 1^3 + 1 = 2$, $y + 2 = 2x + 1$, so $y = 4 + 1 - 2 = 3$.
   The line is thus $y - 3 = 1 * (x - 2)$, or alternatively, you can write the parametric equation $(x, y) = (2, 3) + t(4, 4)$.

2. The cycloid can be parametrized as $x = a(t - \sin t), y = a(1 - \cos t)$. Find the area under one arch (and above x-axis)$(2')$ and the length of one arch. $(2')$
   Ans: Just use the formula $A = \int_{t_1}^{t_2} y(t)x'(t)dt$. We need to find t_1 and t_2. Notice that at the ends of one arch, the y coordinates are both zero. Then letting $a(1 - \cos t) = 0$, you'll have $\cos t = 1$. You can determine that $t_1 = 0$ and $t_2 = 2\pi$.
   Thus $A = \int_0^{2\pi}(a(1 - \cos t)a(1 - \cos t))dt = a^2 \int_0^{2\pi}(1 - \cos t)^2 dt =$
   $a^2 \int_0^{2\pi}(1 - 2\cos t + \cos^2 t)dt = 3\pi a^2$. Notice that $\cos^2 t = \frac{1+\cos(2t)}{2}$ so that you can calculate the third part.

   As for the second question, we just need to evaluate $L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}dt =$
   $\int_0^{2\pi} \sqrt{a^2(1 - 2\cos t + \cos^2 t) + a^2 \sin^2 t}dt = a \int_0^{2\pi} \sqrt{2 - 2\cos t}dt$. Then using $1 - \cos t = 2\sin^2(t/2)$, you'll get the final answer as $a \int_0^{2\pi} 2|\sin(t/2)|dt = 8a$.

3. (a). Find Cartesian coordinate of $P(\sqrt{2}, \pi/4)$ (in polar coordinate). Plot it. $(2')$
   (b). Change the Cartesian equation into its equivalent polar equation: $y^2 = 4x$ $(1')$

   Ans: (a). $x = r\cos\theta = \sqrt{2}\cos(\pi/4) = \sqrt{2} * \frac{\sqrt{2}}{2} = 1$ and $y = r\sin\theta = \sqrt{2}\frac{\sqrt{2}}{2} = 1$. The Cartesian coordinate is $(1, 1)$. Plotting it is easy to do.
   (b). Plugging in $x = r\cos\theta$ and $y = r\sin\theta$, we'll have $(r\sin\theta)^2 = 4r\cos\theta$.
   $r\sin^2\theta = 4\cos\theta$. You can't divide anything now, because they may be zero.

Bonus 1:(a). Fill in the blanks $(3')$:
$\int_2^2 \sqrt{10\cos^2 t + 8\sin^2 t}dt =$ ________ $\qquad \int_{-1}^{1}(1 + \frac{t}{\sqrt{10\cos^2 t + 8\sin^2 t}})dt =$ ________
If s is the **arc length** parameter, then $\int_2^5 \sqrt{x'(s)^2 + y'(s)^2}ds =$ ________
(b). The velocity vector for $x = x(t), y = y(t)$ is $\overrightarrow{v} =< \frac{dx}{dt}, \frac{dy}{dt} >$. Give a geometric explanation for the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. $(2')$

Ans:(a). For the first integral, notice that the lower limit and the upper limit are the same, so the integral must be 0. Actually, it's impossible to get the accurate antiderivative of the integrand.
For the second integral, notice the result of $\int_{-1}^{1} dt = 2$ and the second part is the integral of an odd function, so the answer is 2.
For the third, just notice that the formula is used to calculate the arc length. Thus the answer should be the arc length between $s_1 = 2$ and $s_2 = 5$. However, s is already the arc length parameter, thus the length is $5 - 2 = 3$.

(b). We know the slope is the $\tan\theta$ value where θ is the angle between the tangent line and x-axis. However, the velocity vector is just parallel to the tangent line, thus
$\tan\theta = (dy/dt)/(dx/dt)$

Bonus 2: Polar eqns: What's this curve called $r = \frac{6}{2+3\cos\theta}$?(2′) How about $r = 2\cos\theta$? (1′)

Ans: For the first eqn, $2r + 3r\cos\theta = 6$. Then $2r = 6 - 3x$. Taking square, $4r^2 = 9(2-x)^2$. $4(x^2 + y^2) = 9(x^2 - 4x + 4)$. Then $5x^2 - 36x - 4y^2 + 36 = 0$. The coefficient of x^2 and y^2 are of opposite signs and the set of points are nonempty. Thus it's a hyperbola.
For the second, $r^2 = 2r\cos\theta$, and thus $x^2 + y^2 = 2x$. You'll find that it's a circle.