

# Keys to Quiz1

By Lei January 26, 2011

1. Use integral by parts to evaluate  $\int x^2 e^{-x} dx$  (5 pts)

Ans: Notice  $x^2$  is a polynomial and thus we let  $u = x^2, dv = e^{-x} dx$  first.

$du = 2x dx, v = -e^{-x}$ , and

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int (-e^{-x} 2x) dx = -x^2 e^{-x} + 2 \int x e^{-x} dx.$$

Letting  $u = x, dv = e^{-x} dx$ , we have

$$-x^2 e^{-x} + 2 \int x e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

2.  $\int_0^1 \sqrt{1-z^2} dz$  (3 pts)

Hint: substitution  $z = \sin \theta$

Ans: If  $z = \sin \theta, dz = \cos \theta d\theta$  and  $\sqrt{1-z^2} = \cos \theta$ . The new limits become

$\sin^{-1} 0 = 0$  and  $\sin^{-1} 1 = \pi/2$ . The integral thus becomes:

$$\int_0^{\pi/2} \cos \theta \cos \theta d\theta = \int_0^{\pi/2} \cos^2 \theta d\theta = \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta = \pi/4 + 0 = \pi/4$$

3.  $\int \frac{\sin 2x}{1+\sin x} dx$  (2 pts)

Ans:  $\sin(2x) = 2 \sin x \cos x. \int \frac{\sin 2x}{1+\sin x} dx = \int \frac{2 \sin x \cos x}{1+\sin x} dx.$

Letting  $u = \sin x$ , we have

$$\int \frac{2u}{1+u} du = 2 \int (1 - \frac{1}{1+u}) du = 2u - 2 \ln |1+u| + C = 2 \sin x - 2 \ln(1 + \sin x) + C$$

Bonus 1: Prove  $\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx$  (1pt) and use it to calculate  $\int \tan^3 x dx$  (2 pts)

$$\begin{aligned} \text{Ans: } \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx = \\ &\int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx = \int u^{n-2} du (u = \tan x) - \int \tan^{n-2} x dx \end{aligned}$$

The equation holds.

Hence,

$$\int \tan^3 x dx = \frac{1}{2} \tan^2 x - \int \tan x dx = \frac{1}{2} \tan^2 x - \ln |\sec x| + C = \frac{1}{2} \tan^2 x + \ln |\cos x| + C$$

Bonus 2:  $\int e^{\sin^2 x} \sin(2x) dx$  (2 pts)

$$\text{Ans: } u = \sin^2 x, du = 2 \sin x \cos x dx = \sin(2x) dx. \int e^u du = e^u + C = e^{\sin^2 x} + C$$