Focus on explaining how to solve problems, namely summarize methods, instead of explaining theorems or concepts.

1 Sequences and series:

1). For sequences, basically, we should be able to calculate their limits.
   A. For the limits of fractions, you can either use L’Hôpital’s rule or just pick the
      dominating terms.
   B. $0 \div \infty$ or $\infty \div \infty$, you can use L’Hôpital’s rule. Otherwise, not \( \frac{\sin n}{n} \). Some types can be reduced to these two. $0 \ast \infty$, $0^0 \infty^0$ and $1^\infty$.
      Example, $n^2 \ln(\cos(1/n))$, $n^{\ln 2}/\ln n$
   C. Use Sandwich Theorem. This is usually applied to oscillating sequences.
   D. Use our magic tools. (This is what I call them, so you shouldn’t say this in exam, be other TAs won’t understand by “Magic tools”)
   E. Common limits
      Example: $\lim_{n \to \infty} \frac{\sin(1/n)}{1/n}$, $\lim_{n \to \infty} \frac{n^3-2n^2+4}{3n^3+4n^2-8}$, $\lim_{n \to \infty} \frac{\sin n}{\sqrt{n}}$

Find the limits of the sequence: $a_n = \sqrt[3]{2n+1}$, $a_n = n^2(1-\cos(1/n))$
$a_n = \sqrt[3]{\ln n}$

2). Understand what’s called the sum of the series. Then understand what happens if a series fails or passes $n$-th term test for divergence.
   Example: Does this sequence converge $a_n = \frac{e^n}{e^n+n}$? Does this series $
\sum \frac{e^n}{e^n+n}$ converge?

3). Geometric series and telescoping series are the two kinds of series that we can get the sum. If you see ”get the sum of” in exam, then you should know it is either geometric or telescoping.
   Geometric series must have a constant multiplier. Telescoping can be written as $\sum (f(n) - f(n+1))$
   Example: $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$, $\sum_{n=1}^{\infty} (\tan^{-1}n - \tan^{-1}(n+1))$
   $\sum (\sqrt[n]{n} - \sqrt[n+1]{n+1})$, $\sum \frac{2}{4n^2-1}$
4). $p$-series

5). For determining the convergence and divergence: Maybe you can follow the following steps to determine the convergence or divergence (State the test you are using and check the conditions that must be satisfied to receive full credits):

If the series is **eventually** nonnegative, consider the nonnegative part:

a). $n$-th term test for divergence

b). See if they are geometric or telescoping and you can determine their convergence (for these two, the series doesn’t have to be nonnegative in exam).

c). Use DCT or LCT to compare them with geometric or $p-$series. You may want to pick the dominating terms to find the comparison series to use LCT. You can also use some simple algebra or our 'Magic tools' to find the series to use DCT.

d). Use Integral test to check. You must check that the function is nonnegative, decreasing and continuous.

e). Use Ratio Test.

f). Use $n$-th root test.

If the series is eventually nonpositive, then, move the negative sign out of the summation and go back to the former case.

If the signs are in disorder:

Use the absolute value and return the first case.

Example:

\[ \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{4^n} \]
\[ \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \]
\[ \sum_{n=1}^{\infty} \frac{n^4}{3^n} \]

\[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]
\[ \sum_{n \geq 1} \frac{1}{\sqrt{n} \sqrt{n+1}} \]
\[ \sum_{n=1}^{\infty} \frac{3^n}{n!} \]
\[ \sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \]
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 3^n}}{(16 + 1/n)^{n/2}} \]

2 Improper Integrals

a). See if the interval contains infinity or bad points. If not, it is normal integral. Otherwise it’s improper.

b). See if we can get the antiderivative. If yes, use the definition to get the answer.

c). Use DCT or LCT to solve that problem.
Example: \( \int_0^\infty x^2 e^{-x} \, dx \) \( \int_0^1 \ln x \, dx \) \( \int_0^\frac{1}{\sqrt{2}} \, dx \)

**State the test you are using to receive full credits**

Example: Do the integrals converge?

\( \int_1^\infty \frac{e^{-t}}{\sqrt{t}} \, dt \) \( \int_1^\infty \frac{1}{e^t+4} \, dx \) \( \int_0^x \frac{2x}{x^4+4} \, dx \)

### 3 Integration

#### 3.1 Basic Formulas and substitution

Refer to 8.1. I listed some frequently used below (however others may also be tested):

\[
\int u^\alpha \, du = \frac{1}{\alpha + 1} u^{\alpha + 1} + C (\alpha \neq -1) \quad \int \frac{1}{u} \, du = \ln |u| + C
\]

\[
\int \sec^2 u \, du = \tan u + C
\]

\[
\int \sec u \tan u \, du = \sec u + C \quad \int \tan u \, du = \ln |\sec u| + C
\]

\[
\int a^u \, du = \frac{a^u}{\ln a} + C \quad \int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1}(u/a) + C
\]

\[
\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1}(u/a) + C \quad \int \sec u \, du = \ln |\sec u + \tan u| + C
\]

Example: \( \int \frac{1}{\sqrt{8x-x^2}} \, dx \) \( \int \frac{\ln x}{x + x \ln x} \, dx \) \( \int \frac{4u}{1+4u^2} \, du \) \( \int \sqrt{4y^2+1} \, dy \)

#### 3.2 Integration by parts and reduction formulas

\[
\int u \, dv = uv - \int v \, du
\]

Example: \( \int (x^2 + 2x)e^x \, dx \) \( \int \tan^{-1} x \, dx \) \( \int x \cos(x - 3) \, dx \)
3.3 Integration of rational functions

I won’t list the formulas. Just remind you two things:

1). Fraction must be polynomial over polynomial. If not, you can try whether you can get polynomial over polynomial by substitution.

2). Check whether it’s improper fraction first. If it’s improper, reduce it first.

Example: \[ \int \frac{2x^3 - 4x^2 - 3}{x^2 - 2x - 3} \, dx \]
\[ \int \frac{1}{x^3 + 1} \, dx \]
\[ \int \frac{1}{2x^2 - 3} \, dx \]

Note: Cover-up only applies for linear factors. Multiply and then take special x’s to get coefficients may be fast sometimes.

3.4 Trig integrals and trig substitution

Example: \[ \int \sin^3 x \cos^2 x \, dx \] \[ \int \sec^3 x \, dx \] or \[ \int \tan^2 x \sec x \, dx \]

\[ \sin 2x = 2 \sin x \cos x \]
\[ \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \]

Example: \[ \int \sqrt{1 + \cos 4x} \, dx \]
\[ \sqrt{a^2 + x^2} \quad x = a \tan \theta \]
\[ \sqrt{1 + a^2 x^2} \quad ax = \tan \theta \]
\[ \sqrt{a^2 - x^2} \quad x = a \cos \theta \text{ or } a \sin \theta \]
\[ \sqrt{x^2 - a^2} \quad x = a \sec \theta \]

Example: \[ \int \sqrt{1 - x^2} \, dx \] \[ \int \frac{1}{x^2 \sqrt{4 - x^2}} \, dx \]

3.5 Calculating volumes

If the solid is generated by revolving about one horizontal line, I suggest you using cross-section method.

If revolving about one vertical line, I suggest you using shell method.

Example: Consider the infinite region on the right of \( x = 1 \), between \( y = e^{-x} \) and \( x \)-axis.

a). Calculate the volume of the solid generated by revolving this region about \( x \)-axis.

b). Calculate the volume of the solid generated by revolving this region about \( y \)-axis.

Note: for a) the volume is \( \int_1^\infty \pi e^{-2x} \, dx \) and for b), the volume is \( \int_1^\infty 2\pi xe^{-x} \, dx \)