Consider the function \( f(x) \) defined by the power series:

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n} (x - 1)^n.
\]

a). Determine the interval where \( f(x) \) is defined. (In other words, find the interval of convergence for the power series). (3’)

b). Determine where the series converges absolutely, conv. conditionally and diverges. (2’)

c). Find the power series form for \( f'(x) \) (1 pt) and a closed form for \( f'(x) \) (2’)

d). Find a closed expression for \( f(x) \) (2’)

Ans: a). Apply ratio test or n-th root test. Here, I prefer ratio test.

\[
a_{n+1} \frac{a_n}{a_{n+1}} = \frac{n}{n+1} (x - 1)
\]

So \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - 1| \). Require \( |x - 1| < 1 \) and we have \( 0 < x < 2 \).

Check whether it converges at endpoints. when \( x = 0 \), it is \( \sum (-1)^n \frac{1}{n} \), by AST(some brief justification needed), it converges. However, at \( x = 2 \), it’s harmonic series and thus diverges. So \([0, 2)\)

b). We know it converges on \([0, 2)\) and converges absolutely on \((0, 2)\). It diverges everywhere else. So, we must check what kind of convergence at \( x = 0 \). Consider \( \sum |a_n| \), and you can see it’s just a harmonic series if we sum the absolute values. So it doesn’t converge absolutely and thus converges conditionally.

c). On the interval where it’s defined, we take derivative term by term and have:

\[
f'(x) = \sum_{n=1}^{\infty} \frac{1}{n} n (x - 1)^{n-1} = \sum_{n=1}^{\infty} (x - 1)^{n-1}
\]

Then, you can see it’s geometric series. First term \( a = 1 \), and multiplier is \( r = x - 1 \). It converges only if \( |x - 1| < 1 \) which has the same radius of convergence as \( f(x) \). Then \( f'(x) = \frac{1}{1-(x-1)} = \frac{1}{2-x} \) if \( 0 < x < 2 \).

d). \( f(x) = \int f'(x) dx = -\ln |2-x| + C \). Plug in \( x = 1 \), and according to the power series form, we can know \( f(1) = 0 \). Thus \( -\ln 1 + C = 0 \). \( C = 0 \).

Bonus: We know it’s impossible to get a closed form of \( \int_0^x e^{-t^2} \) dt. However, we can actually get a power series expression for it.

a). Get this power series expression.(3’) (Hint: Get the Maclaurin series of \( e^{-t^2} \))

b). Get an approximation for \( I = \int_0^1 e^{-t^2} \) dt by keeping the first 3 terms in your power series and estimate the error using AST. (2’)

Ans: a). \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) and thus \( e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \).

Then, integrate term by term, \( \int_0^x e^{-t^2} \) dt = \( \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \) \( \frac{t^{2n}}{n!} \) dt = \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \)

b). Plug in \( x = 1 \) and we have \( I = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \). If we use the first three terms \( (n = 0, 1, 2) \), then \( |error| < |(-1)^3| \)