Keys to Quiz3

By Lei  September 22, 2011

1. a). Find the area of the region bounded by $y = \frac{x^2}{\sqrt{9-x^2}}$, $x$-axis and $x = \frac{3}{2}$ (4 pts)

b). Find the volume of the solid generated by revolving the region bounded by $y = \frac{x}{\sqrt{9-x^2}}$, $x$-axis and $x = \frac{3}{2}$ about $y$-axis. (1 pt).

Hint: You can make use of the integral in a).

Ans: a). Draw the graph of the function, you can see that the graph actually passes through the origin and that’s the reason why I didn’t mention ’bounded by $y$-axis and ...’.

The integral is $Area = \int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}} dx$.

All of you know that you should do trig substitution $x = 3 \sin \theta$. However, when you do this substitution, many of you forgot that the bounds need to be changed. However, at present, we don’t want to focus on the bound, so you can do the indefinite integral $\int \frac{x^2}{\sqrt{9-x^2}} dx$ first.

If $x = 3 \sin \theta$, $dx = 3 \cos \theta$.

$\int \frac{x^2}{\sqrt{9-x^2}} dx = \int \frac{9 \sin^2 \theta}{3 \cos \theta} 3 \cos \theta d\theta = 9 \int \sin^2 \theta d\theta = \frac{9}{2} \int (1-\cos(2\theta)) d\theta = \frac{9}{2} (\theta - \frac{1}{2} \sin(2\theta)) + C$

Two ways to solve the definite integral now. One way is to get the antiderivative of the original function, namely substituting back. $\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$ and drawing the triangle, you’ll get the indefinite integral as $\frac{9}{2} (\sin^{-1}(\frac{x}{3}) - \frac{x \sqrt{9-x^2}}{3}) + C$. Then plug in the bounds, we get $\frac{9}{2} (\sin^{-1}(\frac{3}{4}) - \frac{3 \sqrt{3}}{8}) - \frac{9}{2} (0 - 0) = \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}$

Since our goal is to get the definite integral, we don’t have to substitute back. Here, we know when $x = 0$, $\theta = 0$ and when $x = \frac{3}{2}$, $\theta = \frac{\pi}{6}$ and thus the answer is $\frac{9}{2} (\theta - \frac{1}{2} \sin(2\theta))|_0^{\pi/6} = \frac{9}{2} (\frac{\pi}{6} - \frac{1}{2} \sin(\frac{\pi}{3})) = \frac{9}{2} (\frac{\pi}{6} - \frac{\sqrt{3}}{4})$

b). The axis is just $y$-axis, and thus we can use the shell method. The formular is $\int_a^b 2\pi * r * h dx$. Here, the radius is $r = |x - L| = x$ and the height is $h = f(x)$. The volume is thus $V = \int_0^{3/2} 2\pi x \frac{x}{\sqrt{9-x^2}} dx = 2\pi * (\text{integral in a})) = 2\pi (\frac{3\pi}{4} - \frac{9\sqrt{3}}{8})$

Note: Somebody argued that the radius would be $\frac{3}{2} - x$ and they thought the region is empty from 0 to $x$ and only from $x$ to $3/2$ is the region valid. However, you should recall that we are not slicing the region parallel to $x$-axis but rather we’re slicing parallel to $y$-axis. Recall the cylinder generated by revolving, and you can find that the radius is $x$ and whether the region is empty between 0 and $x$ or not doesn’t matter.
2. Maybe, you want to use cover-up, or multiply the denominator and pick special x’s.
   a). \( \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} \, dx \) (3 pts)
   b). \( \int \frac{x^3+4x^2+3x-2}{(x-1)(x+1)(x+3)} \, dx \) (2 pts). Hint: b) is closely related to a).

Ans: a). Notice that the degree of the numerator is less than that of the denominator. Then \( \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3} \). Multiply the denominator on both sides: 

\[
\begin{align*}
   x^2 + 4x + 1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x+1)(x-1) \\
   \text{Letting } x &\to 1, 6 = A * 8 + 0 + 0 \\
   B &= \frac{1}{2}, \quad x \to -3, \quad -2 = 8C, \quad C = \frac{1}{4} \\
   \int \left( \frac{3}{4} \cdot \frac{x}{x-1} + \frac{1}{2} \cdot \frac{1}{x+1} - \frac{1}{4} \cdot \frac{1}{x+3} \right) \, dx &= \frac{3}{4} \ln |x-1| + \frac{1}{2} \ln |x+1| - \frac{1}{4} \ln |x+3| + C
\end{align*}
\]

b). Somebody did the following wrong answer:

\[
\int \frac{x^3+4x^2+3x-2}{(x-1)(x+1)(x+3)} \, dx = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3} \]. Then determine A,B,C. Oh! They are the same with a) and the answer should be the same!

This is wrong because the fraction is improper, and you must use long division to reduce it first. \((x-1)(x+1)(x+3) = (x^2 - 1)(x+3) = x^3 + 3x^2 - x - 3\). Then you can find that \( \frac{x^3+4x^2+3x-2}{(x-1)(x+1)(x+3)} = 1 + \frac{A}{x^2+4x+1} \). At this point, let’s see why it’s wrong. You can see that if you multiply the denominator on both sides, you then pick special values of \(x\), you’ll always kill the constant 1 and you won’t get any information about it. We can conclude generally that if we don’t do long division but instead determine the A, B, C first, we would only get the fraction part and lose the polynomial part!

The remaining is easy now, because the fraction part is the same as a). The answer is

\[
x + \frac{3}{4} \ln |x-1| + \frac{1}{2} \ln |x+1| - \frac{1}{4} \ln |x+3| + C
\]

Bonus: \( \int \frac{1}{(x^2-1)^2} \, dx \). Suppose Lei wanted to do like this:

1st: Ohmm, the denominator is a square of a quadratic form. Write out the general form \( \frac{Ax+B}{x^2-1} + \frac{Cx+D}{(x^2-1)^2} \) and determine A, B, C, D.

2nd: For \( \frac{C}{(x^2-1)^2} \), just do substitution \( u = x^2 - 1 \). For \( \frac{1}{(x^2-1)^2} \), use trig substitution. However, Lei forgot that this quiz shouldn’t contain such integrals. Maybe, he was wrong. Find out why Lei was wrong (1 pt) and give the correct way to do this. (2 pts)

Ans: Lei was wrong because \( x^2 - 1 \) is reducible. If you follow Lei’s step, you’ll get \( A = B = C = 0, D = 1 \), which is obviously incorrect!

Instead, we must do \( \frac{1}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \)