An elementary proof of two formulas in trigonometry

By Lei, Sep. 7th

In the section today, I was asked why $1+\cos(x)=2\cos^2(\frac{x}{2})$ and I wanted to prove $\cos(x)=\cos^2\left(\frac{x}{2}\right)-\sin^2(\frac{x}{2})$. However I was stuck that time. After the section, I immediately realized it was actually very direct. Below, I'll prove *cos(a+b)=cos(a)cos(b)-sin(a)sin(b)* and *sin(a+b)=sin(a)cos(b)+cos(a)sin(b)* for $0 \le a, b \le \frac{\pi}{2}$.

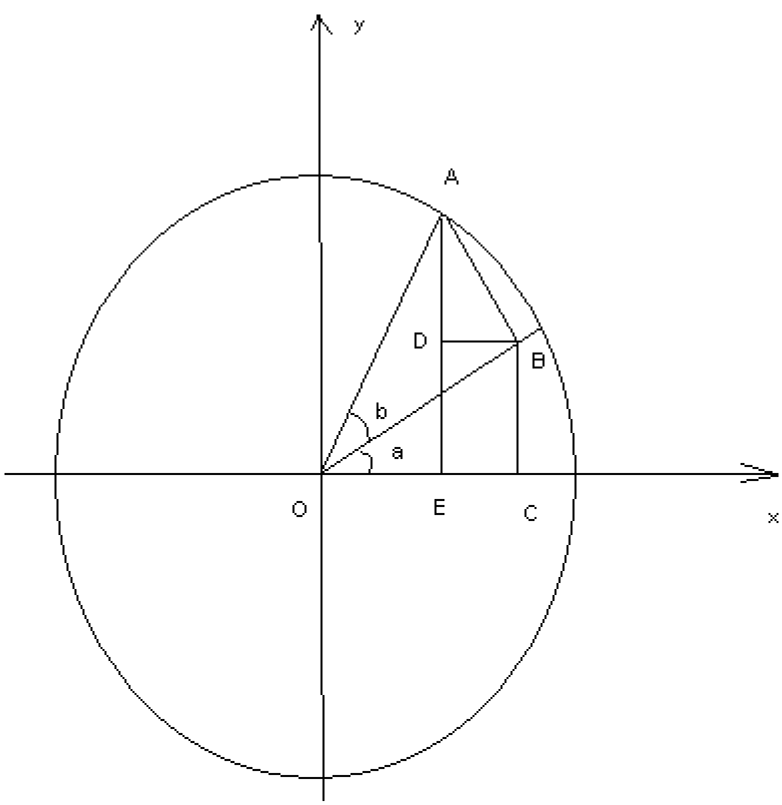
As for the general case, they are just some corollaries of these two basic equations.

A.

First of all, if a or b is equal 0 or pi/2, the equations are obvious correct.

Now let's look at the other cases.

Let's look at the first one:

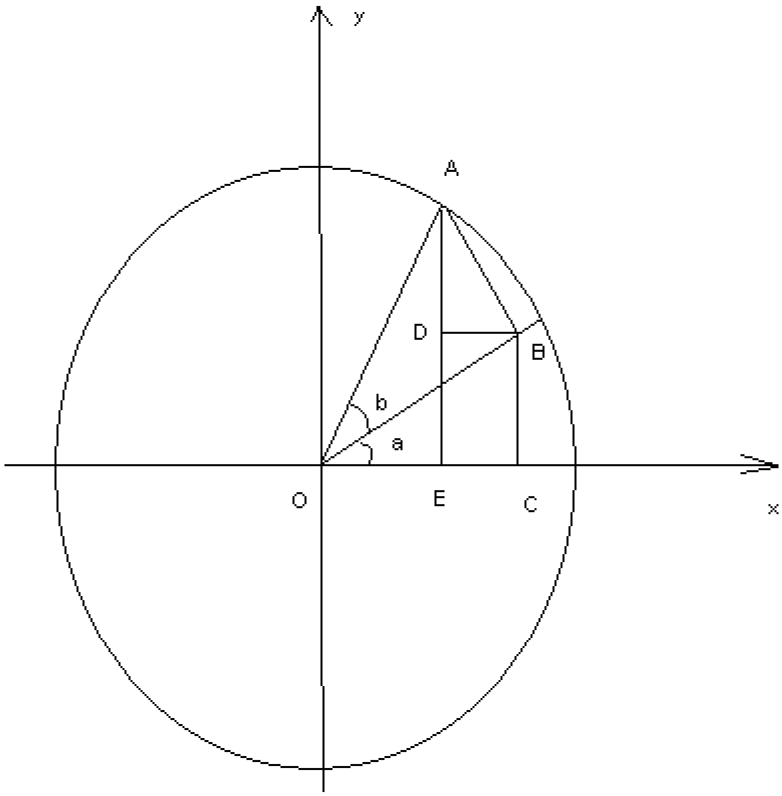


We have unit circle here, so OB=cos(b) and thus OC=OB*cos(a)=cos(b)*cos(a);

EC=BD=AB*sin(angle BAD)=AB*sin(a)=sin(b)*sin(a).

Hence OE=OC-EC, and this is actually cos(a+b)=cos(a)cos(b)-sin(a)sin(b).

Let's look at the second one.



We have AD=AB*cos(BAD)=AB*cos(a)=sin(b)cos(a) and DE=BC=OB*sin(a)=cos(b)*sin(a).

Hence sin(a+b)=AE= DE+AD=sin(a)cos(b)+cos(a)sin(b).

B. For general a and b, we can use that $\sin(x) = \sin(x + 2k\pi)$, $\cos(x) = \cos(x + 2k\pi)$ cos(-x)=cos(x), $\sin(\pi - x) = \sin(x)$, and $\cos(\pi - x) = -\cos(x)$ etc to reduce them to the above cases. We can see that the two equations are also right.

cos(a+b)=cos(a)cos(b)-sin(a)sin(b)

sin(a+b)=sin(a)cos(b)+cos(a)sin(b)

C. By replacing b as –b, we have:

cos(a-b)=cos(a)cos(b)+sin(a)sin(b)

sin(a-b)=sin(a)cos(b)-cos(a)sin(b)

D. We can also have from the above equations that:

cos(a)cos(b)=[cos(a+b)+cos(a-b)]/2

sin(a)sin(b)=[cos(a-b)-cos(a+b)]/2

sin(a)cos(b)=[sin(a+b)+sin(a-b)]/2

cos(a)sin(b)=[sin(a+b-sin(a-b)]/2

E. Let a=b=x, and we have:

$$\cos(2x) = cos^2(x) - sin^2(x) = 2cos^2(x) - 1 = 1 - 2sin^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$