Material for the last discussion

This is a part of the topics, so you **shouldn't** only use this to have the review. Read your book and do the example problems. Do the problems in Midterms and homework once again to have a review.

Some suggestions:

1). Stay until last minute in the final.

2). Do easy problems first. If you can't finish one problem quickly or feel confused, just skip it first and come back later.

3). Check your answer using different ways. For example, differentiate back to check your integral. Plug in the solution to differential equations to check. Check $a \times b$ by taking the dot product of your answer with the two vectors to see whether they are zeros. For power series, you can take particular values of x to see whether the interval of convergence is correct. Use ratio test or root test to check the other one, etc.

4). Ask others and discuss with others if you have questions.

5).Memorize some important conclusions and theorems. For example, some basic integral formulae, integral by parts, the partial fractions, some trigonometric functions/substitution, L'Hopital's rule, p-series, telescoping series, geometric series, conclusions of power series, Taylor series, two methods of error estimation, four types of differential equations, dot product, cross product, distance between a point and a line or a plane, arc length, curvature, etc.

1 Integral

1.1 Basic Formulae

$$\int u^n du = \frac{1}{n+1}u^{n+1} + C \qquad \int \frac{1}{u}du = \ln|u| + C \qquad \int \sec^2 u du = \tan u + C$$

$$\int \sec u \tan u du = \sec u + C \qquad \int \tan u du = \ln|\sec u| + C$$

$$\int a^u du \qquad \int \frac{1}{\sqrt{a^2 - u^2}}du = \sin^{-1}(u/a) + C$$

$$\int \frac{1}{a^2 + u^2}du = \frac{1}{a}\tan^{-1}(u/a) + C \qquad \int \sec u du = \ln|\sec u + \tan u| + C$$

Example: $\int \frac{1}{\sqrt{8x-x^2}}dx \quad \int e^{\tan u}\sec^2 u du \quad \int \frac{4u}{1+4u^2}du$

1.2 Integral by parts

$$\int udv = uv - \int vdu$$

Example: $\int (x^2+2x)e^x dx \quad \int (x^2+2x)\sin x dx \quad \int \cos x e^x dx$

1.3 Integral by partial fractions

Requirements:

1). The integrant must be polynomial over polynomial. If not, you can try whether you can get polynomial over polynomial by substitution.

2). Check whether it's improper fraction first. If it's improper, reduce it first.

Example: $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx \quad \int \frac{1}{x(x^2+1)^2}dx$

Note: Cover-up only applies for linear factors. Multiply and then take special x's to get coefficients may be fast sometimes.

1.4 Trig integral

Two types.

$$\int \sin^m(x)\cos^n(x)dx \qquad \int \sec^m x \tan^n x dx$$

Example: $\int \sin^3 x\cos^2 x dx \quad \int \sec^3 x dx$ or $\int \tan^2 x \sec x dx$

$\sin 2x = 2\sin x\cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Example: $\int \sqrt{1+\cos 4x}dx$

1.5 Trig substitution

$$\sqrt{a^2+x^2} \quad x = a\tan\theta \quad \sqrt{1+a^2x^2} \quad ax = \tan\theta$$
$$\sqrt{a^2-x^2} \quad x = a\cos\theta \ or \ a\sin\theta$$
$$\sqrt{x^2-a^2} \quad x = a\sec\theta$$

Example: $\int \frac{x^2}{9-x^2}dx$

1.6 Improper Integral

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$\int_a^b f(x)dx = \lim_{c\to a}\int_c^b f(x)dx \quad \text{if a is a singular point}$$

Example: $\int_0^\infty x^2e^{-x}dx \quad \int_0^1 \frac{1}{\sqrt{x}}dx$

Tests for convergence

Definition; Direct comparison; Limit Comparison

Example: $\int_0^\infty \frac{1}{e^x+5}dx$ converges? $\quad \int_2^\infty \frac{1}{x(\ln x)^p}dx$

For this kind of problems, you can usually take the dominant terms to see.

2 series

2.1 Sequence

Just some limits. Sandwich, L'Hopital and some other techniques may be used.

Example: Limits. $a_n = \sqrt{n^2+1}/(n+2)$. $a_n = \frac{3\ln(2n+2)}{\ln n^2}$ $a_n = n^3(\frac{1}{n} - \sin(1/n))$

Several important conclusions.

$$(\ln n)^p \ll n^q \ll a^n \ll n! \ll n^n \quad p>0, q>0, a>1$$

$$\sqrt[n]{n}\to 1 \quad \sqrt[n]{x}\to 1, x>0 \quad x^n\to 0\ |x|<1 \quad (1+\frac{x}{n})^n\to e^x, \quad n\to\infty$$

2.2 Three important series

Geometric, p-series, telescoping

$$\sum_{n=1}^\infty ar^{n-1} = \frac{first}{1-ratio} = \frac{a}{1-r}, |r|<1$$

Example: $\sum_{n=5}^\infty \frac{(-2)^n 7^{2n}}{(8e)^{3n}} \quad \sum_1^\infty \frac{1}{n(n+1)}$ $\sum_1^\infty \frac{1}{n^{1.1}} \quad \sum_1^\infty(\tan^{-1}(n+1) - \tan^{-1}(n))$

2.3 Tests for convergence

Integral test, Direct Comparison Test, Limit Comparison Test, Ratio Test, Root Test, AST, Absolute convergence test

Example: $\sum_1^\infty \frac{1}{n(1+\ln^2 n)}$

Comparison test, we can usually take take the dominant term to find the comparison series.

Example: $\sum \frac{\sqrt{n}}{n^2+1} \quad \sum \frac{1}{n\sqrt[n]{n}}$

Example: $\sum \frac{2^n}{n^2} \quad \sum (\frac{1}{n+1})^n$

Example: $\sum \frac{(-1)^n}{\sqrt{n+\sqrt{n}}}$. Converge conditionally or absolutely? Error estimation.

2.4 Power series

Radius of convergence, center, Interval of convergence, points where converges absolutely, conditionally, etc

Example: $\sum\limits_1^\infty \frac{(-1)^{n+1}(2x+4)^n}{n2^n}$

2.5 Taylor and Maclaurin series

Two methods, one way is to use formula

$$f(x) = \sum_0^\infty \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$f(x) = \sum_0^N \frac{f^{(n)}(a)(x-a)^n}{n!} + R_N(x), \quad R_N(x) = \frac{f^{(N+1)}(c)(x-a)^{N+1}}{(N+1)!}$$

The other way is to use some known expansions or integrate or differentiate known expansions.

Example: 2^x, $x^2\sin(x^3)$, $\int \tan^{-1}(x)dx$

2.6 Taylor polynomial and two kinds of error estimation

The formula is as above. One way is to use Taylor theorem. Only for alternating can we use AST.

Example: $\sin x = x - x^3/3! + error$
Note: Euler's identity $e^{ix} = \cos x + i \sin x$

3 Differential equations

3.1 First order

separable; linear

$$\frac{dy}{dx} = G(x)H(y)$$

Put all y's on left hand side and all x's on the right hand side, and integrate.

Example: $2dx + y^2 dx = dy/(x^3 + e^x)$

Linear: First of all, reduce it to standard form and then find the integrating factor $e^{\int p}$

Example: $3xy' - y = \ln x + 1$

Application: Tank

3.2 Second order

We only talk about linear equations with constant coefficients.

Homogeneous

$$ay'' + by' + cy = 0$$

The corresponding auxiliary equation $ar^2 + br + c = 0$. Three cases. Example: $y'' + 4y' + 9y = 0 \quad y'' + 2y' + y = 0, y(0) = 0, y'(0) = 2$

Inhomogeneous

General solution is $y_c + y_p$. Using boundary conditions to determine constants

Several cases: e^{rx}, try Ae^{rx} if r is not a root, Axe^{rx} if r is a single root and Ax^2e^{rx} if it is a double root.

polynomial, try polynomial

$\sin(rx)$, try $A \sin rx + B \cos rx$ if ir is not a root, $Ax \sin rx + Bx \cos rx$ if ir is a single root, and $Ax^2 \sin rx + Bx^2 \cos rx$ if ir is a double root. Equivalently, you can check whether $\sin rx$ and $x \sin rx$ is the homogeneous solutions.

Example: $y'' - y = e^x, y(0) = 1, y'(0) = 0$

Application: Spring

4 Vectors

4.1 Sphere

Example: $x^2 + 4x + y^2 - 6y + z^2 + 2z = 0$

4.2 Length and direction, dot product, angle and projection

Example: Length and direction $P(-3, 4, 1), Q(-5, 2, 2), \overrightarrow{PQ}$

$$a, b. Proj_b a = \frac{a \cdot b}{|b|^2} b$$
$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

Example: $u = 6i + 3j + 2k, v = i - 2j - 2k$. Write u as components, one is perpendicular to v and one is parallel to v. Find angle between them.

4.3 Cross product, triple scalar product, area of triangles, volume

We know that $\overrightarrow{PQ} \times \overrightarrow{PR} = |PQ| * |PR| * \sin\theta * \hat{n}$, where $\hat{n}$ is a unit vector which is perpendicular to the plane determined by P, Q, R and satisfies the right hand principle if we go from $\overrightarrow{PQ}$ to $\overrightarrow{PR}$. The area of the triangle should be $\frac{1}{2}|PQ| * |PR| \sin\theta$. The area is $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

2D:

$$Area = \frac{1}{2} \text{abs} \left(\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Here, $u =< u_1, u_2 >= \overrightarrow{PQ}$ and $v =< v_1, v_2 >= \overrightarrow{PR}$

How about the area of the parallelogram? $|a \times b|$

Triple scalar product:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{1}$$

Attention: No i, j, k here! Volume=|scalar product|

Example: Quiz just three weeks ago.

4.4 Equations of lines, Planes

Both need a point on it and a vector.The line needs a vector parallel to it and the plane need a vector perpendicular to it.

$$P(x_0, y_0, z_0), v =< a, b, c >, n =< A, B, C >$$

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct \qquad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Example. Quiz two weeks ago.

4.5 Distance between a point and a line, distance between a point and a plane, distance between two planes or a line and a plane

$$Distance(P, line) = |\frac{\overrightarrow{PS} \times v}{|v|}|$$

$$Distance(P, plane) = |\frac{\overrightarrow{PS} \cdot n}{|n|}|$$

The other two kinds of distances can be reduced to the distance between a point and a plane.

4.6 Angle between two planes, relationship between lines, the line where two planes intersect, line segment

$$\theta = \cos^{-1}(\frac{|n_1 \cdot n_2|}{|n_1||n_2|})$$

For the line where two planes intersect, since the line is on both planes, it should be perpendicular to both normal vectors and thus be parallel to $n_1 \times n_2$, which can be regarded as the vector parallel to it. Finding a point on the line is finding a (only one is enough) solution to both equations of the two planes.

For the line segment, I just want to remind you to check the interval of the parameter t.

Example: Just learned, omitted.

5 Vector functions

5.1 Velocity, acceleration, speed

We just need to differentiate and integrate. Example, quiz.

5.2 Arc length, unit tangent vector, unit normal vector, curvature

$$s = \int_{t_0}^{t} |v(\tau)| d\tau$$

$$T = \frac{dr}{ds} = \frac{dr/dt}{|v|} = v/|v|$$

$$\kappa = |dT/ds| = \frac{1}{|v|}|\frac{dT}{dt}| = \frac{|v \times a|}{|v|^3}$$

$$N = \frac{1}{\kappa}\frac{dT}{ds} = \frac{dT/dt}{|dT/dt|}$$

$$\text{Radius of curvature :}\rho = \frac{1}{\kappa}$$

$$\text{Center of curvature :}O = P + \rho N$$

Here P is on the curve. Then, you can write out the circle of curvature.

Example: $r(t) = \cos 2ti + \sin 2tj + 4k$. Find the arc parameter, the distance along the curve from $t = -\pi$ to $t = \pi$. Find the unit tangent line, curvature, unit normal vector and circle of curvature at $t = \pi/2$