Keys to Quiz7

By Lei October 28, 2010

1. $f(x) = \sqrt{x+1}$. Find the Taylor Polynomial of order 2 $P_2(x)$ (3 terms) generated by f at $x = 0$. Estimate the error when we use $P_2(0.1)$ to calculate $\sqrt{1.1}$ (5 pts)
   Ans: Generally, the Taylor polynomial of order 2 can be denoted as $P_2(x)$ or $T_2f(x)$. Here I choose the former. The general formula for Taylor polynomial of order n is $\sum\limits_{k=0}^{n} \frac{f^{(n)}(a)(x-a)^k}{k!}$. In this problem, we just need to calculate
   $P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$. We have $f(x) = \sqrt{x+1}$, $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ and $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$. (Actually, we have $f^{(n)}(x) = \frac{(-1)^{(n-1)}(2n-3)!!}{2^n}(1+x)^{-(2n-1)/2}$, where $(-1)!! = 1$ and $(2k+1)!! = 1 * 3 * 5 * \ldots * (2k+1)$). We have $f(0) = 1$, $f'(0) = 1/2$ and $f''(0) = -1/4$. Hence, $P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$.
   $P_2(0.1) = 1 + 0.05 - 0.125 * 0.1^2$, which is not required in this problem.
   The error term is $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{(n+1)}}{(n+1)!}$. Here, we have $R_2(0.1) = \frac{f'''(c)0.1^3}{3!} = \frac{1}{16*1000*\sqrt{1+c}^5}$, where c is between 0 and 0.1. We can have $\frac{1}{16*1000*\sqrt{1.1}^5} < |error| < \frac{1}{16*1000*\sqrt{1+0}^5} = \frac{1}{16000}$. Actually, only the upper bound is important, so if you got the upper bound correctly, it's already OK.
   Some students noticed that the series was alternating except the first term and used the upcoming term to estimate. This is also OK, but the result is a little worse than using the Taylor theorem.

2. (a). Find $\cos(i)$ (Euler's identity) and r in "$re^{i\frac{\pi}{3}} = 1 + i\sqrt{3}$", where $i = \sqrt{-1}$ (2 pts)
   Ans: From the Euler's identity, we can deduce that $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ for any complex number z. Hence we have $\cos(i) = \frac{e^{i*i}+e^{-i*i}}{2} = \frac{e+e^{-1}}{2} > 1$. We can see that if the variable is imaginary, the cosine function can be bigger than 1. For the second, we have $r\cos(\pi/3) = 1$ and $r\sin(\pi/3) = \sqrt{3}$, so $r = 2$. Another method is that $|re^{i\pi/3}| = |1 + i\sqrt{3}|$, which implies $r = |1 + i\sqrt{3}| = \sqrt{1^2 + \sqrt{3}^2} = 2$
   (b). Solve the differential equation $\frac{dy}{dx} = 3x^2e^{-y}$ (3 pts)
   Ans: $\int e^y \mathrm{d}y = \int 3x^2 \mathrm{d}x$, so we have $e^y = x^3 + C$ or $y = \ln(x^3 + C)$

3. Bonus.(3 pts)
   Use Taylor Theorem to calculate:
   $\lim\limits_{x \to 0} \frac{(e^x-1-x)(\sin x - x)}{\sin^5 x}$ (3 pts)
   We have $e^x - 1 - x = \frac{x^2}{2!} + x^3(\frac{1}{3!} + smallterms)$ and
   $\sin x - x = -\frac{x^3}{3!} + x^5(-\frac{1}{5!} + smallterms)$. $(\sin x)^5 = (x + x^3(-\frac{1}{3!} + smallterms))^5$. We have:

$$\lim_{x\to 0}\frac{(e^x-1-x)(\sin x-x)}{\sin^5 x}=\lim_{x\to 0}\frac{(\frac{x^2}{2!}+x^3(\frac{1}{3!}+smallterms))(-\frac{x^3}{3!}+x^5(-\frac{1}{5!}+smallterms))}{(x+x^3(-\frac{1}{3!}+smallterms))^5}=$$
$$\lim_{x\to 0}\frac{(\frac{1}{2!}+x(\frac{1}{3!}+smallterms))(-\frac{1}{3!}+x^2(-\frac{1}{5!}+smallterms))}{(1+x^2(-\frac{1}{3!}+smallterms))^5}=\frac{(1/2)(-1/3!)}{1}=-\frac{1}{12}$$