Keys to Quiz5

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1. Choose **ANY ONE** of the two below.(2 points)
   - Which one is zero? (Please Circle.)

   A. $\lim\limits_{n\to\infty}\frac{n}{(\ln n)^4}$ B. $\lim\limits_{n\to\infty}\frac{100n^3}{e^{n/2}}$ C. $\lim\limits_{n\to\infty}\frac{n!}{2^n}$ D. $\lim\limits_{n\to\infty}\frac{n^n}{n!}$

   Ans: B.

   As for the speeds of going to infinity:$(\ln n)^p << (n)^q << a^n(a>1) << n! << n^n$

   - When does $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ converge? (Circle one.)

   A. $p\geq 0$ B. $p<0$ C. $p\geq 1$ D. $p>1$

   Ans: D. Integral Test.

2. 1). Find the limit of **ANY ONE** of the two from Homework 5. (2 pts)

   $a_n = (\frac{1}{n})^{1/(\ln n)}$ $a_n = \sqrt[n]{n^2}$

   Ans: You can choose any one to do.

   $(\frac{1}{n})^{1/(\ln n)} = e^{\ln(\frac{1}{n})^{1/(\ln n)}} = e^{\frac{1}{\ln n}(-\ln n)} = e^{-1}$

   $\lim\limits_{n\to\infty}\sqrt[n]{n^2} = 1$, because $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$.

   2). Find the **sum** of the Geometric Series from Homework 5:

   $\sum\limits_{n=0}^{\infty}(\frac{2^{n+1}}{5^n})$ (3 pts)(Pay attention to the sub-index)

   Ans:Pay attention to the subindex. It's from 0, so the first term is 2. The ratio is obviously 2/5. The answer is $2/(1-2/5) = 10/3$

   Use the Integral Test to decide when the series from Homework 5 converges:

   $\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ (3 pts)

   Ans: No matter what p is, the terms decrease **eventually**, and the terms are all positive. You can use the integral test.

   Use another variable to do the integral please!

   $\int_2^{\infty}\frac{1}{x(\ln x)^p}\mathrm{d}x = \lim\limits_{b\to\infty}\int_2^b\frac{1}{x(\ln x)^p}\mathrm{d}x$

   When $p=1$, it's $\lim\limits_{b\to\infty}\ln(\ln(x))|_{\ln 2}^{\ln b}$, so it diverges.

   When $p\neq 1$, letting $u=\ln x$, it's

   $\lim\limits_{b\to\infty}\int_2^b\frac{1}{x(\ln x)^p}\mathrm{d}x = \lim\limits_{b\to\infty}\int_{\ln 2}^{\ln b}u^{-p}\mathrm{d}u = \lim\limits_{b\to\infty}\frac{1}{1-p}u^{1-p}|_{\ln 2}^{\ln b}$

   If we want it to converge, the term $(\ln b)^{1-p}$ must be cancelled. Hence, $1-p<0$, namely $p>1$. We conclude that the series converges if and only if $p>1$

(Bonus) Choose **ANY ONE** of the three below. (3pts)

- Does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ converge or not? (2 pts)

Ans: Diverges. I have said that for the speed of going to infinity, the natural log is much slower than the power of n. We could expect **eventually** $\sqrt{n}\ln n$ is smaller than $\sqrt{n}n^{1/4} = n^{3/4}$, or equivalently $\sqrt{n}\ln n < n^{3/4}$ eventually. The latter is a p-series with $p = 3/4$, which diverges, so the former diverges.

You can justify this rigorously by using the limit comparison test by using the limit $\lim_{n\to\infty} \frac{\sqrt{n}\ln n}{n^{3/4}}$

How about $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ $(0 < p \leq 1)$? (1 pt)

Similar argument goes to this. When $p < 1$, we have $(p+1)/2$ between p and 1. $n^p \ln n < n^{(p+1)/2}$ eventually. Use the limit comparison test to justify this rigorously by $\lim_{n\to\infty} \frac{1/(n^{(p+1)/2})}{1/(n^p \ln n)} = 0$. $\sum \frac{1}{n^{(p+1)/2}}$ diverges. It diverges. When p=1, use the integral test.

- Does $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converge or not? (2 pts)

Ans:We know the natural log is slower than the power as go to infinity. So $(\ln n)^2/n \to 0$. However, $\sum \frac{1}{n^2}$ converges. Hence, the original series converges.

How about $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^p}$ $(p > 1, q > 1)$? (1 pt)

Ans:Similar argument goes to this. $(\ln n)^q/n^{(p-1)/2} \to 0$ and $\sum \frac{1}{n^{(p+1)/2}}$ converges. Hence $\sum \frac{(\ln n)^q}{n^p} = \sum \frac{(\ln n)^q}{n^{p+1)/2} n^{(p-1)/2}}$ converges.

- Does $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \ldots + \frac{1}{(2k-1)^2} + \frac{1}{(2k)^3} + \ldots$ converge? (1 pt)

Ans: Converges.It's smaller than p-series when p=2.

$\lim_{n\to\infty} \frac{\sin n}{n}$ (1 pt)

Ans: Sandwich theorem says it's 0 because $-1/n \leq a_n \leq 1/n$

$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n}$ (1 pt)

Ans: 1. Using the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ or using L'H here.