1. Choose **ANY ONE** of the two below. (2 points)
   • Which one is zero? (Please Circle.)
   A. \( \lim_{n \to \infty} \frac{n}{(\ln n)^4} \)  
   B. \( \lim_{n \to \infty} \frac{100n^3}{e^{n/2}} \)  
   C. \( \lim_{n \to \infty} \frac{n!}{n^2} \)  
   D. \( \lim_{n \to \infty} \frac{n^n}{n!} \)
   Ans: B.
   As for the speeds of going to infinity:
   \((\ln n)^p \ll (n)^q \ll a^n (a > 1) \ll n! \ll n^n\)
   • When does \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converge? (Circle one.)
   A. \( p \geq 0 \)  
   B. \( p < 0 \)  
   C. \( p \geq 1 \)  
   D. \( p > 1 \)
   Ans: D. Integral Test.

2. 1). Find the limit of **ANY ONE** of the two from Homework 5. (2 pts)
   \( a_n = \left(\frac{1}{n}\right)^{1/(\ln n)} \)  
   \( a_n = \sqrt[n]{n} \)
   Ans: You can choose any one to do.
   \( \left(\frac{1}{n}\right)^{1/(\ln n)} = e^{\ln(\frac{1}{n})^{1/(\ln n)}} = e^{\frac{1}{\ln n}(-\ln n)} = e^{-1} \)
   \( \lim \sqrt[n]{n^2} = 1, \) because \( \lim_{n \to \infty} \sqrt[n]{n} = 1. \)
   2). Find the sum of the Geometric Series from Homework 5:
   \( \sum_{n=0}^{\infty} \frac{2^{n+1}}{n^{(\ln n)^p}} \) (3 pts)
   Pay attention to the sub-index
   Ans: Pay attention to the sub-index. It’s from 0, so the first term is 2. The ratio is obviously 2/5. The answer is \( 2/(1 - 2/5) = 10/3 \)
   Use the Integral Test to decide when the series from Homework 5 converges:
   \( \sum_{n=2}^{\infty} \frac{1}{n((\ln n)^p)} \) (3 pts)
   Ans: No matter what \( p \) is, the terms decrease eventually, and the terms are all positive. You can use the integral test.
   Use another variable to do the integral please!
   \( \int_{2}^{\infty} \frac{1}{x(\ln x)^p} \)  
   When \( p = 1 \), it’s \( \lim_{b \to \infty} \ln((\ln x))_{ln2}^{lnb} \), so it diverges.
   When \( p \neq 1 \), letting \( u = \ln x \), it’s
   \( \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x(\ln x)^p} \)  
   If we want it to converge, the term \((\ln b)^{1-p} \) must be cancelled. Hence, \( 1 - p < 0 \), namely \( p > 1 \). We conclude that the series converges if and only if \( p > 1 \)
(Bonus) Choose ANY ONE of the three below. (3pts)

• Does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n \ln n}}$ converge or not? (2 pts)

Ans: Diverges. I have said that for the speed of going to infinity, the natural log is much slower than the power of n. We could expect eventually $\sqrt{n \ln n}$ is smaller than $\sqrt{nn^{1/4}} = n^{3/4}$, or equivalently $\sqrt{n \ln n} < n^{3/4}$ eventually. The latter is a p-series with $p = 3/4$, which diverges, so the former diverges.

You can justify this rigorously by using the limit comparison test by using the limit $\lim_{n \to \infty} \frac{\sqrt{n \ln n}}{n^{3/4}}$

How about $\sum_{n=2}^{\infty} \frac{1}{n^{p \ln n}}$ (0 < $p$ ≤ 1)? (1 pt)

Similar argument goes to this. When $p < 1$, we have $(p + 1)/2$ between p and 1. $n^p \ln n < n^{(p+1)/2}$ eventually. Use the limit comparison test to justify this rigorously by $\lim_{n \to \infty} \frac{1/(n^{(p+1)/2})}{1/(n^{p \ln n})} = 0$. $\sum n^{1/(n^{(p+1)/2})} diverges. It diverges. When p=1, use the integral test.

• Does $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^q}$ converge or not? (2 pts)

Ans: We know the natural log is slower than the power as go to infinity. So $(\ln n)^2/n \to 0$. However, $\sum \frac{1}{n}$ converges. Hence, the original series converges.

How about $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^p}$ (p > 1, $q > 1$)? (1 pt)

Ans: Similar argument goes to this. $(\ln n)^q/n^{(p-1)/2} \to 0$ and $\sum \frac{1}{n^{(p+1)/2}}$ converges.

Hence $\sum \frac{(\ln n)^q}{n^p} = \sum \frac{(\ln n)^q}{n^{(p+1)/2}}$ converges.

• Does $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(2k-1)^2} + \frac{1}{(2k)^2} + \ldots$ converge? (1 pt)

Ans: Converges. It’s smaller than p-series when p=2.

$\lim_{n \to \infty} \frac{\sin n}{n}$ (1 pt)

Ans: Sandwich theorem says it’s 0 because $-1/n \leq a_n \leq 1/n$

$\lim_{n \to \infty} \frac{\sin(1/n)}{1/n}$ (1 pt)

Ans: 1. Using the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$ or using L’H here.