Math 222 Keys to Quiz4

By Lei   September 30, 2010

---

1. Choose **ANY ONE** of the two below.(3 points)

   • If $\int_1^{+\infty} \frac{1}{x^2}\mathrm{d}x$ means $\lim\limits_{b\to+\infty} \int_1^b \frac{1}{x^2}\mathrm{d}x$, what does $\int_0^{+\infty} \frac{1}{x^2}\mathrm{d}x$ mean? Does it converge? Why?
   Ans:
   Since $\frac{1}{x^2}$ goes to infinity as x goes to 0, we actually have:
   $\int_0^{+\infty} \frac{1}{x^2}\mathrm{d}x = \lim\limits_{a\to0^+} \int_a^1 \frac{1}{x^2}\mathrm{d}x + \lim\limits_{b\to+\infty} \int_1^b \frac{1}{x^2}\mathrm{d}x = \lim\limits_{a\to0^+} (-1+\frac{1}{a}) + \lim\limits_{b\to+\infty} (-\frac{1}{b}+1)$
   The second is 1, but the first is positive infinity. Hence, this integral diverges.

   • Find the error below and give out the correct answer:
   $\int_{-1}^1 \frac{1}{x^{4/3}}\mathrm{d}x = (-3x^{-1/3})|_{-1}^1 = -3+3(-1)^{-1/3} = -6.$
   Ans:
   The function goes to infinity when x goes to 0, so this is actually an improper integral and you can't write it as a definite integral as in this problem.
   The correct answer is:
   $\int_{-1}^1 \frac{1}{x^{4/3}}\mathrm{d}x = \int_{-1}^0 \frac{1}{x^{4/3}}\mathrm{d}x + \int_0^1 \frac{1}{x^{4/3}}\mathrm{d}x = \lim\limits_{a\to0^-} \int_{-1}^a \frac{1}{x^{4/3}}\mathrm{d}x + \lim\limits_{b\to0^+} \int_b^1 \frac{1}{x^{4/3}}\mathrm{d}x =$
   $\lim\limits_{a\to0^-} (-3a^{-1/3}+3(-1)^{-1/3}) + \lim\limits_{b\to0^+} (-3+3b^{-1/3}) = +\infty - 3 - 3 + (+\infty) = +\infty$

2. Calculation. (7 points)

   (a). $\int_0^\infty ye^{-y}\mathrm{d}y$ (3 pts)
   Ans:
   ye^{-y} is continuous on $[0,+\infty)$, so this improper integral is actually
   $\lim\limits_{b\to+\infty} \int_0^b ye^{-y}\mathrm{d}y = \lim\limits_{b\to+\infty} (-ye^{-y} - e^{-y})|_0^b$, where you can use integral by parts to get this.
   It's $\lim\limits_{b\to+\infty} (-be^{-b} - e^{-b} + 0 + 1) = 1$. Many students have difficulty getting $\lim\limits_{b\to+\infty} (be^{-b})$. Please refer to Math 221.

   (b) i). Given $2, 7, 12, 17, 22, \ldots$ where each number except the first one minus its previous one is a constant, find the expression for the n-th term a_n. (2 pts)
   Ans:
   Many students can't get this correctly. A pity.
   $a_1 = 2$
   $a_2 = 2 + 5$
   $a_3 = a_2 + 5 = 2 + 5 * 2$

$a_4 = a_3 + 5 = 2 + 5 * 2 + 5 = 2 + 5 * 3$

...

$a_n = a_{n-1} + 5 = 2 + (n-2) * 5 + 5 = 2 + (n-1) * 5 = 5n - 3.$

ii). Find the limit $\lim\limits_{n\to\infty} \frac{a_n}{\sqrt{n^2+1}}$ (2 pts)

$\lim\limits_{n\to\infty} \frac{a_n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \frac{5n-3}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \frac{5-3/n}{\sqrt{1+1/n^2}} = 5$

(Bonus) 2pts.

If $a_1 = 1$ and $a_{n+1} = a_n/(n+1)$, find $\lim\limits_{n\to\infty} 2^n a_n$

Ans:

$a_2 = a_1/2 = 1/2$

$a_3 = a_2/3 = 1/(2 * 3)$

$a_4 = a_3/4 = 1/(2 * 3 * 4)$

...

$a_n = a_{n-1}/n = \frac{1/(2*3*\ldots*(n-1))}{n} = 1/(2 * 3 * \ldots * n) = 1/n!$

$\lim\limits_{n\to\infty} 2^n a_n = \lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$

If we denote $b_n = 2^n/n!$, we can see that b_n is positive and after $n \geq 3$, it decreases. Hence it must have a limit. Suppose it is b. We have $b_n = b_{n-1} * \frac{2}{n}$. As $n \to \infty$, it is $b = b * 0$, so the limit is 0.

Another way is to see that $b_n \leq (2/3)^{n-3} b_3$ when $n \geq 3$.