Keys to Quiz2

By Lei   September 16, 2010

Your Name: *Your Section:*

Instructions: Time is 20 minutes and the total score is 10 points. You can answer in attached papers if you like. There are extra problems on the back, whose points will be regarded as bonus.

1. Choose **ANY ONE** of the two below.(3 points)
   • Write out the formula for integral by parts and give an example for $\int f(x)g(x)\mathrm{d}x \neq \int f(x)\mathrm{d}x \cdot \int g(x)\mathrm{d}x$.
   Ans: Either $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$ or $\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\mathrm{d}x$ is right.
   Letting $f(x) = x$ and $g(x) = x$, we have $\int x^2\mathrm{d}x = \frac{x^3}{3} + C \neq \int x\mathrm{d}x \cdot \int x\mathrm{d}x = (\frac{x^2}{2} + C)^2$
   • Write out the general expression of partial fractions for $\frac{g(x)}{(x-a)^3(x-b)^2(x-c)}$, where $\deg(g) < 6$ and a, b, c are not equal to each other.
   Ans: $\frac{A}{(x-a)^3} + \frac{B}{(x-a)^2} + \frac{C}{x-a} + \frac{D}{(x-b)^2} + \frac{E}{x-b} + \frac{F}{x-c}$

2. Find the integrals below. (7 points)
   a). $\int_0^1 (y^2 - 2y + 1)e^y\mathrm{d}y$ (4 pts)   b). $\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\mathrm{d}x$ (3 pts)
   Ans:
   a). $u = (y-1)^2, e^y dy = dv$ and $u = y-1, e^y dy = dv$. We can get the antiderivative is $(y-1)^2e^y - 2(y-1)e^y + 2e^y + C$. Thus, we have
   $\int_0^1 (y^2 - 2y + 1)e^y\mathrm{d}y = ((y-1)^2e^y - 2(y-1)e^y + 2e^y)|_0^1 = 2e - 5$
   b). The partial fraction expression for $\frac{x^2+1}{(x-1)(x-2)(x-3)}$ is $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. We can determine that $A = 1, B = -5, C = 5$.
   (We have two ways. The first is Cover-up method and the second one is like this: $x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$. Then you can let x go to 1, 2 or 3)
   So we have:
   $\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\mathrm{d}x = \int (\frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3})\mathrm{d}x = \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$.
   Or, you can write like this: $\ln|\frac{(x-1)(x-3)^5}{(x-2)^5}| + C$

3. Extra problem.(Bonus)

1). Explain why there is a constant term C_1 on the right from (a) to (b). (1 pt)

$\int e^x \cos x \mathrm{d}x = e^x \sin x - \int e^x \sin x \mathrm{d}x = e^x \sin x + e^x \cos x - \int e^x \cos x \mathrm{d}x \quad$ (a)

$2\int e^x \cos x \mathrm{d}x = e^x \sin x + e^x \cos x + C_1 \quad$ (b)

Ans: Because the antiderivative is the set of functions of the form $F(x) + C$. The set $\int e^x \cos x \mathrm{d}x$ plus the set $-\int e^x \cos x \mathrm{d}x$ is the set of constants by recalling the sum of two sets is the set of the sums of the elements. Thus we have C_1 be the set of the constants.

2). Find the integral. (3 pts)

$\int \frac{x^2 \sin(\ln x) + (2\sin(\ln x) + 3)x + \sin(\ln x) - 1}{x^2 + 2x + 1} \mathrm{d}x$

Ans: $x^2 \sin(\ln x) + (2\sin(\ln x) + 3)x + \sin(\ln x) - 1 = 3x - 1 + (x^2 + 2x + 1)\sin(\ln x)$, so we have:

$\int \frac{x^2 \sin(\ln x) + (2\sin(\ln x) + 3)x + \sin(\ln x) - 1}{x^2 + 2x + 1} \mathrm{d}x = \int \frac{3x-1}{(x+1)^2} \mathrm{d}x + \int \sin(\ln x) \mathrm{d}x$

$= \int \frac{3}{x+1} \mathrm{d}x - \int \frac{4}{(x+1)^2} \mathrm{d}x + \int \sin(\ln x) \mathrm{d}x = 3\ln|x+1| + \frac{4}{x+1} + \frac{1}{2}[x\sin(\ln x) - x\cos(\ln x)] + C$

Note:

The numerator is not a polynomial, so you can't write it as $\frac{A}{(x+1)^2} + \frac{B}{x+1}$

The optional exercise:

$\int \frac{1}{x^2+1} \mathrm{d}x = \frac{x}{x^2+1} + \int \frac{2x^2}{(x^2+1)^2} \mathrm{d}x = \frac{x}{x^2+1} + \int \frac{2}{x^2+1} \mathrm{d}x - 2\int \frac{1}{(x^2+1)^2} \mathrm{d}x$

So we have $\int \frac{1}{(x^2+1)^2} \mathrm{d}x = \frac{1}{2}\tan^{-1}(x) + \frac{x}{2(x^2+1)} + C$.

Some students do as $u = \tan x$, and this may also work.

For $\int \frac{1}{(x^2+bx+c)^2} \mathrm{d}x$, when $b^2 - 4c > 0$, the integral is like $\int \frac{1}{(x-x_1)^2(x-x_2)^2} \mathrm{d}x$. When $b^2 - 4c = 0$, it's like $\int \frac{1}{(x-x_0)^4} \mathrm{d}x$. When it's negative, we go back to something like $\int \frac{1}{(x^2+1)^2} \mathrm{d}x$.