Keys to Quiz12

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1. A(1, 2, 3). Two planes $P_1 : x + y = 1$ and $P_2 : 2x + y - 2z = 2$. Find the angle between planes (2'), the parametrizations of the line where the planes intersect (2') and the distance between $A$ and $P_2$ (1').

Ans: Nomal vector of $P_1$ is $n_1 = <1, 1, 0>$ and normal vector of $P_2$ is $n_2 = <2, 1, -2>$. The angle is thus $\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) = \pi/4$.

The line should be parallel to $n_1 \times n_2 = <-2, 2, -1>$. We need to find a point on the line. Any point satisfying $x + y = 1$ and $2x + y - 2z = 2$ should work. I choose $P(1, 0, 0)$. The line is $x = 1 - 2t, y = 2t, z = -t$.

Distance $d = |\frac{\overrightarrow{AS \cdot n_2}}{|n_2|}|$. $S$ is a point on $P_2$. If we choose $S(1,0,0)$. We have $d = |\begin{pmatrix} 0 \ -2 \ -3 \end{pmatrix} - \begin{pmatrix} 1 \ 0 \ 2 \end{pmatrix}| = 4/3$.

2. (a). $r(t) = \cos 2t \mathbf{i} + 3 \sin 2t \mathbf{j} + 4 \mathbf{k}$. Find the velocity, the speed and acceleration at $t = \pi$ (3')

(b). $\frac{dr(t)}{dt} = \sec t \tan t \mathbf{i} + \tan 2t \mathbf{j} + 2 \sin t \cos t \mathbf{k}$. $r(0) = \overrightarrow{0}$. Find $r(\pi/3)$ (2')

Ans: (a). $v(t) = \frac{dr(t)}{dt} = -2 \sin 2t \mathbf{i} + 6 \cos 2t \mathbf{j}$ and $a(t) = \frac{dv(t)}{dt} = -4 \cos 2t \mathbf{i} - 12 \sin 2t \mathbf{j}$. At $t = \pi$, we have $v(\pi) = 6j$ and speed is $|6j| = 6$. $a(\pi) = -4i$.

(b). $r(\pi/3) = r(0) + \int_0^{\pi/3} \sec t \tan t \mathbf{i} + \tan 2t \mathbf{j} + 2 \sin t \cos t \mathbf{k} dt = i + \ln 2j + 3k/4$. Here $\int \sec t \tan t dt = \sec t + C$, $\int \tan t dt = \ln |\sec t| + C$ and $2 \sin t \cos t = \sin 2t$.

(Bonus) A particle is moving. If $v \cdot \frac{dv}{dt} = 1$ and the speed at $t = 0$ is 0, find the speed at $t = 2$ (2')

Furthermore, let it be on $y^2 = 2x$. If at $t = 2$, it’s at $(2, 2)$ and moving from left to right, find the velocity (2').

Ans: Since $v \cdot \frac{dv}{dt} = \frac{1}{2} \frac{d|v|^2}{dt} = 1$, we have $|v|^2 = 2t + c$. We can determine that $c = 0$. Then $v(2) = 2$.

At $(2, 2)$, one vector parallel to the tangent line is $<1, f'(x)> = <1, 1/\sqrt{2x}> |_2 = <1, 1/2>$. The velocity is thus $2*<1, 1/2> * = <4, 2>/\sqrt{5}$.