Keys to Quiz11

By Lei December 2, 2010

1. (a).$P(1,1,1), Q(2,1,3), R(3,-1,1)$. Find the area of the triangle determined by these three points and one unit vector perpendicular to this triangle. If the points are $P(1,1), Q(2,1), R(3,-1)$, find the area again.$(4')$
   Ans: We know that $\overrightarrow{PQ} \times \overrightarrow{PR} = |PQ| * |PR| * \sin\theta * \hat{n}$, where $\hat{n}$ is a unit vector which is perpendicular to the plane determined by P, Q, R and satisfies the right hand principle if we go from $\overrightarrow{PQ}$ to $\overrightarrow{PR}$. The area of the triangle should be $\frac{1}{2}|PQ| * |PR| \sin\theta$. We conclude that the area is $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = i \begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} = 4i + 4j - 2k \quad (1)$$

The area is thus $Area = \frac{1}{2}\sqrt{4^2 + 4^2 + (-2)^2} = 3$.
The unit vector is $\frac{1}{6}(4i + 4j - 2k) = 2i/3 + 2j/3 - k/3$
One way to check your cross product is to see whether your result dot those two vectors is zero!
For the triangle in the two dimensional space, you can deduce the the formula is:

$$Area = \frac{1}{2}\text{abs}\left(\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right) \quad (2)$$

Here, $u =< u_1, u_2 >= \overrightarrow{PQ}$ and $v =< v_1, v_2 >= \overrightarrow{PR}$. You can get this formula by letting $P(1,1,0), Q(2,1,0), R(3,-1,0)$. Anyway the area is:

$$Area = \frac{1}{2}\text{abs}\left(\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}\right) = \frac{1}{2} * 2 = 1. \quad (3)$$

(b). $\mathbf{u} = 2\mathbf{i} + \mathbf{j}, \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{w} = \mathbf{i} + 2\mathbf{k}$. Find the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ and the volume of the parallelepiped determined by these three vectors.$(2')$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = -4 - 3 + 0 = -7 \quad (4)$$

The volume is the absolute value of the sacalar product and hence is $volume = |-7| = 7$
Note: Some students wrote like this:

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = -4i - 3j + 0(\textbf{wrong!}) \tag{5}$$

I just want to ask these students: how do i and j come out?

2. $P(3,-2,1)$ L: $x = 1+2t, y = 2-t, z = 3t$. Find the line through P and parallel to L $(2')$, the distance beween P and L $(2')$, and the plane through P and perpendicular to L $(2')$.
   Ans: A vector which is parallel to the line can be found directly by taking the coefficients of t. $v =< 2,-1,3 >$. The new line is parallel to L and hence parallel to v. The equation thus is $x = 3+2t, y = -2-t, z = 1+3t$

   The distance between P and L can be calculated like this: find a point S on L. Then, $d = |PS| * \sin\theta = |\overrightarrow{PS} \times v|/|v|$. Letting $t = 0$, we can find a point on the line $S(1,2,0)$. The distance is thus
   $| < -2,4,-1 > \times < 2,-1,3 > |/| < 2,-1,3 > | = | < 11,4,-6 > |/\sqrt{14} = \sqrt{\frac{173}{14}}$

   The plane is perpendicular to L and thus perpendicular to v. Then v can be regarded as the normal vector $n =< A,B,C >= v =< 2,-1,3 >$. The equation is thus $2*(x-3) - (y+2) + 3(z-1) = 0$, which is $2x - y + 3z = 11$

3. Choose **THREE** of the following. Circle directly. No need to explain.If you answer more than three, I'll grade the first three you answered.$(3')$
   ①. If $\angle AOB = \pi/3$, the angle between $\overrightarrow{AO}$ and $\overrightarrow{OB}$ is: $(\pi/3$ vs $2\pi/3)$
   Ans: When we want to see the angle, we usually move the vectors to make their initial points the same and see. By this way, we can see that the angle should be $2\pi/3$

   ②. Is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ equal to $\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$: (Yes vs No)
   Ans: We can switch them counter-clockwise to keep the same scalar product. Hence $a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$. We can see now the answer is No.

   ③. If $\mathbf{u}\cdot\mathbf{v_1} = \mathbf{u}\cdot\mathbf{v_2}$, is $\mathbf{v_1} = \mathbf{v_2}$ right: (Yes vs No)
   Ans: No. This is your homework. For example
   $u =< 1,0 >, v_1 =< 1,0 >, v_2 =< 1,1 >$. They have the same projection onto u doesn't imply they are equal.

   ④. If the relation in ③ is right for all $\mathbf{u}$, what's the answer: (Yes vs No)
   Ans:Yes. This implies that v_1 and v_2 have the same projection onto any direction.

⑤. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is always the volume of the parallelepiped determined by the three vectors even if the determinant may be negative: (True vs False)
Ans: False. Look at 1(b), which is a good example.

⑥. $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2$ equals: ($|\mathbf{a} + \mathbf{b}|^2$ vs $|\mathbf{a}|^2|\mathbf{b}|^2$)
Ans: $|a|^2|b|^2$. You can recall that $a \times b = |a| * |b| * \sin\theta * \hat{n}$ and $a \cdot b = |a| * |b| \cos\theta$