Section 11.9
1. Ans: Applying \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) here, we can get \( e^{-5x} = \sum_{n=0}^{\infty} \frac{(-5x)^n}{n!} \)

6. Ans: \( \cos(x^{3/2}/\sqrt{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n(x^{3/2}/\sqrt{2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (2n)!} \)

9. Ans: \( \frac{x^2}{2} - 1 + \cos x = \frac{x^2}{2} - 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \)

13. Ans: \( \cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2} \left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2^n (2n)!} \)

14. Ans: Similar to above \( \sin^2 x = \frac{1}{2} \left(1 - \cos 2x \right) = \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2^n (2n)!} \)

18. Ans: \( \frac{2}{(1-x)^3} \). You can calculate the derivatives directly. Here I’d like to do like this:

\[
\left( \frac{1}{1-x} \right)^n = \left( \sum_{n=0}^{\infty} x^n \right)^n = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n
\]

21. Ans: By the Taylor theorem, we use \( P_2 \) here, we have \( \sin x = x + R_2 = x + \frac{-\cos(c)}{3!} x^3 \). \( |error| < \frac{10^{-3}}{6} \). If \( x < \sin x \), we have \( R_2 > 0 \), which means \( x < 0 \).

22. Ans: We have the formula \( \sqrt{1 + x} = 1 + x/2 + f^{(2)}(c) x^2 / 2^2 \). We can calculate that \( f''(x) = -\frac{1}{4} (1+x)^{-3/2} \), so we have \( |error| = \left| \frac{1}{8(1+c)^{3/2}} x^2 \right| = \frac{1}{8(1-0.01)^{3/2}} x^2 \approx \frac{1}{0.01^{3/2}} x^2 < \frac{1}{0.01^{3/2}} x^2 \). If you would like to do \( e < 3 \), it’s OK, but I don’t think this is good enough.

23. Ans: The error term is \( e^x x^3/3! \), so we have \( |error| < e^{0.1} (0.1^3)/6 \). If you would like to do \( e < 3 \), it’s OK, but I don’t think this is good enough.

28. Ans: We have the formula \( \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \) for \( |x| \leq 1 \). You can get this for \( |x| < 1 \) by integrating \( \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \) and then extend to \( x = \pm 1 \) by the continuity.

Anyway, we have \( \frac{x}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \), which is a convergent alternating series. If we keep \( N \) terms, namely, we use \( \sum_{n=0}^{N-1} \frac{(-1)^n x^{2n+1}}{2n+1} \), then we have \( |error| < \frac{1}{2N+1} \) by the conclusion of alternating series. If we want to keep 2 good decimal places, which means in \( 0.a_1 a_2 \ldots, a_1 \) and \( a_2 \) should be good. Here I’m not quite clear what \( good \) should mean. I understand it by using the approximation \( 0.1194 \approx 0.199 \) and \( 0.1195 \approx 0.120 \), namely more than or equal to 5 contributes 1 to the previous digit and no contribution otherwise. Here the third digit is not good. \( \frac{1}{2N+1} < 5 \cdot 10^{-4} \). I think you can use \( < 10^{-5} \) here.

35. Ans: Not hard but needs a little calculation. I’d like to omit it here. I just give the forth term as an example. We should decide the coefficient of \( x^5 \) (Attention: The term begins with \( x \), and the coefficient of \( x^4 \) is 0, so the fourth term is \( x^5 \)).
1 * \(\frac{1}{3!} + \frac{1}{4!} \cdot (-1)\frac{1}{3!} + \frac{1}{4!} \cdot 1 = -\frac{1}{30}\). You can also take the imaginary part of \(e^{(1+i)x}\).

36. Ans: Similar to the above. Omitted.

Section 9.1

1, 2, 6–8,9–14, 17, 19–22, 23, 24

1 and 2 can be done by differentiation. I’d like give 2 as an example: \(y = -\frac{1}{x+c}\), so
\[y' = -(-1)\frac{1}{(x+c)^2} (x + c)' = y^2\]

6 to 8 can be done directly. I’d like to take 8 as an example. Check the initial value first:
\(y(e) = e/\ln e = e\), which is correct. Then the equation: \(y' = \frac{1}{\ln x} - \frac{1}{(\ln x)^2}\) and you can see \(x^2y' = xy - y^2\) is right.

9–14 and 17 are all separable equations.

9. Ans: \(\sqrt{y}dy = \frac{1}{2\sqrt{x}} dx\). Then, integrate, and we have \(\frac{2}{3}y^{3/2} = \sqrt{x} + C\)

10. Ans: \(2\sqrt{y} = \frac{1}{3}x^3 + C\)

11. Ans: \(e^ydy = e^x dx\), so we have \(e^y = e^x + C\)

12. Ans: \(e^ydy = 3x^2 dx\). Then integrate, and we have \(\int e^ydy = \int 3x^2 dx\), which is \(e^y = x^3 + C\)

13. Ans: Firstly, we should separate the variables. We have \(\frac{dy}{\sqrt{y\cos^2\sqrt{y}}} = dx\). Then integrate, and we have \(\int \frac{1}{\sqrt{y\cos^2\sqrt{y}}}dy = \int dx\). On the left hand side, we do the substitution \(u = \sqrt{y}\). We can have \(2\arctan(\sqrt{y}) = x + C\)

14. Ans: \(\frac{3}{2}x^{3/2} = \sqrt{2x} + C\)

17. Ans: \(\arcsin(y) = x^2 + C\), which is \(y = \sin(x^2 + C)\)


23 and 24 are omitted.