Math 222 Keys and Hints for HW10
By Lei November 11, 2010

I HAVE NO ANSWERS AND THE FOLLOWINGS ARE WHAT I GOT. I FOUND THE CALCULATION WAS NOT FUN AND I MIGHT MAKE MISTAKES. BE CAREFUL WHEN YOU READ WHAT I WROTE.

Section 17.2 3,4,8,9,12,29,30,31,52,58

For an inhomogeneous second order, linear differential equation with constant coefficients, the general solution can always be written as any particular solution $y_p$ plus the general solution to the complementary equation $y_c$ that is the corresponding homogeneous equation.

3. Ans: $y'' - y' = \sin x$. The complementary equation is $y'' - y' = 0$ and the auxiliary equation is $r^2 - r = 0$. $y_c = C_1 e^x + C_2$. $\pm i$ is not the root, so we can try $y_p = A \sin x + B \cos x$. $y_p = \frac{1}{2} (\cos x - \sin x)$. The general solution is $y(x) = C_1 e^x + C_2 + \frac{1}{2} (\cos x - \sin x)$

4. Ans: Complementary equation $y'' + 2y' + y = 0$ and $y_c = C_1 e^{-x} + C_2 xe^{-x}$. Since the coefficient of $y$ is not zero, we can try $y_p = Ax^2 + Bx + C$, and then we can decide $y_p = x^2 - 4x + 6$. The general solution $y(x) = C_1 e^{-x} + C_2 xe^{-x} + x^2 - 4x + 6$

8. Ans: The corresponding complementary equation $y'' + y = 0$. $y_c = C_1 \cos x + C_2 \sin x$. We have two force terms on the right hand side. We can find $y_p$ one by one (Why? Justify yourself! Maybe I’ll give this as a bonus problem in quiz). For 2$x$ we only need 2$x$. For 3$e^x$, since 1 isn’t a root of the auxiliary equation, we can use $Ae^x$ to try and get $3e^x/2$. We finally get $y(x) = C_1 \cos x + C_2 \sin x + 2x + 3e^x/2$.

9. Ans: The complementary equation $y'' - y = 0$. $y_c = C_1 e^x + C_2 e^{-x}$. For $x^2$, we have $-x^2 - 2$. Since 1 is a single root of the auxiliary equation, we can try $Ax e^x$ and we get $xe^x/2$. Finally, we have $y(x) = C_1 e^x + C_2 e^{-x} - x^2 - 2 + xe^x/2$

12. Ans: Complementary: $y'' + 3y' + 2y = 0$. $y_c = C_1 e^{-x} + C_2 e^{-2x}$. For $-x$, we can try $Ax + B$ and we get $-\frac{1}{2} x + \frac{3}{4}$. Since both -2 and -1 are single roots, we can try $A_1 x e^{-x}$ and $A_2 x e^{-2x}$ respectively. We can have $xe^{-x}$ and $-xe^{-2x}$. $y(x) = C_1 e^{-x} + C_2 e^{-2x} + xe^{-x} - xe^{-2x}$

29. Ans: Complementary $y'' - 5y' = 0$. $y_c = C_1 e^{5x} + C_2$. Since 5 is a single root and the force term is $xe^{5x}$, we can try $Ax^2 e^{5x} + Bxe^{5x}$.

$y'' = 2Ae^{5x} + 20Axe^{5x} + 25Ax^2 e^{5x} + 10Be^{5x} + 25Bxe^{5x}$.

$y_p = 2Axe^{5x} + 5Ax^2 e^{5x} + Be^{5x} + 5Bxe^{5x}$. $y_p = \frac{1}{16} x^2 e^{5x} - \frac{1}{25} x e^{5x}$

$y = C_1 e^{5x} + C_2 + \frac{1}{16} x^2 e^{5x} - \frac{1}{25} x e^{5x}$

30. Ans: $C: y'' - y' = 0$. $y_c = C_1 e^x + C_2$. Since $\pm i$ is not the root and the right hand is the sin, cos with frequency 1, we can try $A \sin x + B \cos x$ directly. We have $-\sin x$. $y = C_1 e^x + C_2 - \sin x$

31. Ans: $C: y'' + y = 0$. $y_c = C_1 \cos x + C_2 \sin x$. Now, since $\pm i$ are the roots. For sin and cos on the right, we must use $x$ multiplying the corresponding term. Namely, we should try
\[ y_p = Ax \cos x + Bx \sin x. \ y_p = -\frac{1}{2}x \cos x + x \sin x. \] The general solution is
\[ y = C_1 \cos x + C_2 \sin x - \frac{1}{2}x \cos x + x \sin x \]
52. Ans: \( y_c = C_1 \cos x + C_2 \sin x \).
2 is not the root and we have \( y_p = e^{2x} \). Hence, the general solution \( y = C_1 \cos x + C_2 \sin x + e^{2x} \). We then use the initial values to determine the coefficients. \( y(0) = 0 \) implies \( C_1 = -1/5 \). \( y'(x) = -C_1 \sin x + C_2 \cos x + 2e^{2x} \). We have \( C_2 + 2/5 = 2/5, \) so \( C_2 = 0 \). Finally, we have \( y(x) = -1/5 \cos x + e^{2x} \)
58. Ans: You must verify yourself that the given expression is the particular solution. I’ll omit it here. \( y_c = C_1 e^x + C_2 x e^x \) and hence the general solution is \( y(x) = C_1 e^x + C_2 x e^x + x e^x \ln x \). We then use the two values to determine the coefficients.
\( y(1) = C_1 e + C_2 e + 0 = e, \) which means \( C_1 + C_2 = 1 \).
\( y'(x) = (C_1 + C_2) e^x + C_2 x e^x + e^x \ln x + x e^x \ln x + x e^x \). \( y'(1) = (C_1 + 2C_2) e + e = 0, \) which means \( C_1 + 2C_2 = -1 \). We have \( y(x) = 3e^x - 2xe^x + xe^x \ln x \)

**Section 17.3**

1, 7, 8, 22

1. Ans: Set positive direction to be downward. Let \( y \) be the displacement away from the equilibrium. We can ignore the gravity if we consider the displacement from the equilibrium, since \( mg = ks \) as in Sec 17.3. The instantaneous velocity is \( dy/dt \), and the resistance, or the friction is \(-dy/dt\) as the problem says. The resultant force is \(-y - dy/dt\).

\( mg = G \) implies \( m = G/g = 16/32 = 1/2 \) lb \( \cdot \) sec\(^2 \) ft \( = 1/2 \) slug. Newton’s law says:
\[ \frac{1}{2} \frac{d^2y}{dt^2} = -y - \frac{dy}{dt} \]
\( y(0) = 2 \) and \( y'(0) = 2 \)

7. Ans: The problem is the same as 1. Here, we must solve this equation:
This is a homogeneous second order differential equation with constant coefficients.
\[ \frac{1}{2} \frac{d^2y}{dt^2} + r + 1 = 0. \] Hence \( y(t) = e^{-t}(C_1 \cos t + C_2 \sin t) \) ft.
\( C_1 = 2 \) and \( C_2 = 4 \). \( y(\pi) = e^{-\pi}(-2) < 0 \), so the mass is above the equilibrium and the distance is \( 2e^{-x} \) ft.

8. Ans: Since \( mg = ks \), we know the spring constant is \( k = 8 \) lb/4 ft = 2 lb/ft.
\( m = mg/g = 8/32 = 1/4 \) slug. Also let the positive direction be downward and \( y \) be the displacement from equilibrium. We can have the equation:
\[ \frac{1}{4} y'' = -2y - 1.5y', \] namely \( y'' + 6y' + 8y = 0 \).
\( y(0) = -2 \) (above) and \( y'(0) = 3 \).
\( y(t) = C_1 e^{-2t} + C_2 e^{-4t} \).
\( y(t) = -\frac{5}{2} e^{-2t} + \frac{1}{2} e^{-4t} \) and \( y(2) = \ldots \)

22. Ans: The equation can be written as \( 10y'' = -140y - 90y' + 5 \sin t \). \( y(0) = 0 \) and \( y'(0) = -1 \)
y = \( C_1 e^{-2t} + C_2 e^{-7t} + \frac{13}{500} \sin t - \frac{9}{500} \cos t \). Then you calculate the following, and I’d like to stop here.