Calculus & Analytic Geometry II — Series Packet
Math 327 - 330 / *circle your section* — Fall 2009

Name:________________________

1. Which of these series converge and which diverge? For each series, write *why* it either converges or diverges. Use reasons like "It is the tail of a given series" or "it is a geometric series" or "diverges by the no way test".

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{4n^2-1}$

(e) $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$

(f) $\sum_{n=1}^{\infty} \frac{\cos^4(\tan^{-1}(n))}{n\sqrt[4]{n}}$

(g) $\sum_{n=1}^{\infty} e^n n^{-3}$

(h) $\sum_{n=1}^{\infty} e^{-2n}$

(i) $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

(j) $\sum_{n=1}^{\infty} \frac{3}{n+4}$

(k) $\sum_{n=0}^{\infty} \frac{1}{n!}$

2. Find a closed form (Taylor series) for the following.

   (a) $1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}x^3 + \ldots$

   (b) $5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \frac{(5x)^7}{7!} + \ldots$

   (c) $1 - x^3 + x^6 - x^9 + \ldots$

3. Each of the following series is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

   (a) $1 - \frac{2}{3} + \frac{2}{9} - \frac{4}{81} + \ldots + (-1)^n \frac{2^n}{n!3^n} + \ldots$

(b) $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \ldots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \ldots$

4. Here are the first few terms of the Taylor series around 0 of the tangent of x. It converges when $-\pi/2 < x < \pi/2$:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots$$

Find the first five terms in the series for $\sec^2(x)$ and for $\ln|\sec(x)|$. What is the radius of convergence for each of them?

5. Find the Taylor series for $\sin^2(x)$ and $\cos^2(x)$. (Hint: Use double angle formulas)

6. Write the Taylor series for $\frac{1}{1+x^2}$.

(a) Use the previous Taylor series to find the Taylor series for $\tan^{-1}(x)$.

(b) Give an exact value for the following series:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

7. Observe that $\frac{1}{2+x} = \frac{1}{2(1+(x/2))} = \frac{(1/2)}{1+(x/2)}$. Therefore we may write $\frac{1}{2+x} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{-x}{2}\right)^n$. Repeat this process to find a Taylor series for $\frac{7}{5-x}$:

   (a) Centered around $x = 0$

   (b) Centered about $x = 2$

8. Use the definition of a Taylor series to find the Taylor series for the following functions:

   (a) $\ln(1 + x)$ at $a = 0$.

   (b) $\cos(x)$ at $a = \pi/4$.

9. Which of the following statements are true?

   (a) If $a_n \geq 0$ for every n, then $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

   (b) If $a_n \geq 0$ for every n, then $\sum_{n=1}^{\infty} na_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

   (c) If $a_n \geq 0$ for every n and $a_{n+1} \leq a_n$, and there exists a positive number c such that $a_n \geq c$ for every n, then $\{a_n\}$ converges.

10. Find the Maclaurin series for the following functions:

   (a) $\frac{e^x+e^{-x}}{2}$

   (b) $\frac{x^2}{1+x}$

   (c) $x \cos(\pi x)$

   (d) $e^x - (1 + x)$

11. What happens if you add a finite number of terms to a convergent series? A divergent series? What happens if you multiply a convergent series by a nonzero constant? A divergent series?

12. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series of nonnegative numbers, what can be said about $\sum_{n=1}^{\infty} a_n b_n$?

13. If $a_n \geq 0$ for every n and $\sum_{n=1}^{\infty} a_n$ converges, what can you say about the series $\sum_{n=1}^{\infty} \frac{a_n}{a_n + 1}$?