Calculus & Analytic Geometry II — Series Packet Solutions
Math 327 - 330 / *circle your section* — Fall 2009

Name:________________________

1. Which of these series converge and which diverge? For each series, write *why* it either converges or diverges. Use reasons like "It is the tail of a given series" or "it is a geometric series" or "diverges by the no way test".

   (a) $\sum_{n=1}^{\infty} \frac{1}{2n^3}$ This series converges as it is a constant times a p-series having $p > 1$.

   (b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ The series diverges by the n-th term test; $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.

   (c) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ As $n^2+1 > n^2$, we know that $\sqrt{n^2+1} > \sqrt{n^2}$ and $n\sqrt{n^2+1} > n\sqrt{n^2}$, making $\frac{1}{n\sqrt{n^2+1}} < \frac{1}{n\sqrt{n^2}}$. This series is therefore term by term smaller than the p-series with $p = 2$, which converges; hence this series converges.

   (d) $\sum_{n=1}^{\infty} \frac{6}{4n^2-1}$ Rewriting $\frac{6}{4n^2-1}$ as $\frac{6}{(2n-1)(2n+1)}$, we use partial fractions. The fraction breaks up as $\sum_{n=1}^{\infty} \frac{3}{2n-1} - \frac{3}{2n+1}$, which is a telescoping series with $s_n = 3 - \frac{3}{2n+1}$, and thus $\lim_{n\to\infty} s_n = 3$.

   (e) $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ This is the tail of the geometric series $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$. As $\frac{e}{\pi} < 1$, the series converges.

   (f) $\sum_{n=1}^{\infty} \frac{\cos^4(\tan^{-1}(n))}{n\sqrt[4]{n}}$ As $0 \leq \cos^4(y) \leq 1$ for ANY value of y, we have that this series is term by term less than or equal to the p-series having $p = \frac{5}{4} > 1$, so the series converges.

   (g) $\sum_{n=1}^{\infty} e^n n^{-3}$ This series diverges by the nth term test; $\lim_{n\to\infty} \frac{e^n}{n^3} = \infty \neq 0$.

   (h) $\sum_{n=1}^{\infty} e^{-2n}$ This is a tail of the geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{e^2}\right)^n$, which has radius $r = \frac{1}{e^2} < 1$, so the series converges.

(i) $\sum_{n=0}^{\infty}(-1)^n\frac{5}{4^n}$ If we rewrite the series as $\sum_{n=0}^{\infty}5\left(\frac{-1}{4}\right)^n$, we see that we have a convergent geometric series with $|r|<1$.

(j) $\sum_{n=1}^{\infty}\frac{3}{n+4}$ This series is a constant multiple of a tail of the harmonic series, so it diverges.

(k) $\sum_{n=0}^{\infty}\frac{1}{n!}$ For $n \geq 4$, we know that $n! \geq 2^n$, so this series is term by term less than or equal to $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ except for a finite number of terms; thus the series converges.

2. Find a closed form (Taylor series) for the following.

(a) $1+\frac{1}{3}x+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}x^2+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}x^3+\dots$
$1+\sum_{n=1}^{\infty}\binom{(1/3)}{k}x^k$

(b) $5x-\frac{(5x)^3}{3!}+\frac{(5x)^5}{5!}-\frac{(5x)^7}{7!}+\dots\ \sum_{n=0}^{\infty}\frac{(-1)^n(5x)^{2n+1}}{(2n+1)!}$

(c) $1-x^3+x^6-x^9+\dots\ \sum_{n=0}^{\infty}(-x)^{3n}$

3. Each of the following series is the value of the Taylor series at $x=0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

(a) $1-\frac{2}{3}+\frac{2}{9}-\frac{4}{81}+\dots+(-1)^n\frac{2^n}{n!3^n}+\dots\ e^{-2/3}$

(b) $1-\frac{\pi^2}{9\cdot 2!}+\frac{\pi^4}{81\cdot 4!}-\dots+(-1)^n\frac{\pi^{2n}}{3^{2n}(2n)!}+\dots\ \cos(\pi/3)=\frac{1}{2}$

4. Here are the first few terms of the Taylor series around 0 of the tangent of x. It converges when $-\pi/2<x<\pi/2$:

$$\tan(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\dots$$

Find the first five terms in the series for $\sec^2(x)$ and for $\ln|\sec(x)|$. What is the radius of convergence for each of them? Both series have the same radius of convergence as the series for $\tan x$, which is $\pi/2<x<\pi/2$. The first few terms of the series are:

$$\sec^2(x)=1+x^2+\frac{2x^4}{3}+\frac{17x^6}{45}+\frac{62x^8}{315}$$
$$\ln|\sec x|=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\frac{17x^8}{2520}+\frac{31x^{10}}{14175}$$

5. Find the Taylor series for $\sin^2(x)$ and $\cos^2(x)$. (Hint: Use double angle formulas) Use the double angle formulas $\sin^2(x) = \frac{1-\cos(2x)}{2}$. Write out the Maclaurin series for $\cos(x)$ and write one for $\cos(2x)$ using substitution. Then take $1 - \cos(2x)$ by canceling out the ones and changing the $(-1)^n$ in the formula for $\cos(2x)$ to a $(-1)^{n+1}$. Then divide by 2 to get $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$. Do something similar for $\cos^2(x)$; it becomes $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!}$

6. Write the Taylor series for $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

   (a) Use the previous Taylor series to find the Taylor series for $\tan^{-1}(x)$. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

   (b) Give an exact value for the following series:

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

   $\tan^{-1}(1) = \pi/4$.

7. Observe that $\frac{1}{2+x} = \frac{1}{2(1+(x/2))} = \frac{(1/2)}{1+(x/2)}$. Therefore we may write $\frac{1}{2+x} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{-x}{2}\right)^n$.

   Repeat this process to find a Taylor series for $\frac{7}{5-x}$:

   (a) Centered around $x = 0$ $\frac{7}{5-x} = \frac{(7/5)}{1-x/5} = \sum_{n=0}^{\infty} \frac{7}{5}\left(\frac{x}{5}\right)^n$

   (b) Centered about $x = 2$ $\frac{7}{5-x} = \frac{7}{5-x+2-2} = \frac{7}{5-(x-2)-2} = \frac{7}{3-(x-2)} = \frac{(7/3)}{1-(\frac{x-2}{3})}$ So we get $\sum_{n=0}^{\infty} \frac{7}{3}\left(\frac{(x-2)}{3}\right)^n$

8. Use the definition of a Taylor series to find the Taylor series for the following functions:

   (a) $\ln(1+x)$ at $a = 0$. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

   (b) $\cos(x)$ at $a = \pi/4$. NOTE: I meant this problem to say to compute only the first four terms, which are $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{2\sqrt{2}}(x - \pi/4)^2 + \frac{1}{6\sqrt{2}}(x - \pi/4)^3 + \dots$

9. Which of the following statements are true?

   (a) If $a_n \geq 0$ for every n, then $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \sum_{n=1}^{\infty} \sqrt{a_n}$ converges. This is false; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series, but $\sum_{n=1}^{\infty}$ is the harmonic series, which clearly diverges.

   (b) If $a_n \geq 0$ for every n, then $\sum_{n=1}^{\infty} na_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. This is true; the terms na_n are term by term larger than the terms a_n.

   (c) If $a_n \geq 0$ for every n and $a_{n+1} \leq a_n$, and there exists a positive number c such that $a_n \geq c$ for every n, then $\{a_n\}$ converges. This is also true; this is the definition of a monotone bounded series.

10. Find the Maclaurin series for the following functions:

    (a) $\frac{e^x+e^{-x}}{2}$ $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

    (b) $\frac{x^2}{1+x}$ $\sum_{n=0}^{\infty} x^{n+2}$

    (c) $x\cos(\pi x)$ $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!}$

    (d) $e^x - (1+x)$ $\sum_{n=2}^{\infty} \frac{x^n}{n!}$

11. What happens if you add a finite number of terms to a convergent series? A divergent series? What happens if you multiply a convergent series by a nonzero constant? A divergent series?

    None of these operations alter the convergence or divergence or a series. This is the concept behind the tail of a series.

12. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series of nonnegative numbers, what can be said about $\sum_{n=1}^{\infty} a_n b_n$? As $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, we know the terms are approaching zero, and thus at some point become smaller than one. Therefore $a_n b_n \leq a_n$, and the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

13. If $a_n \geq 0$ for every n and $\sum_{n=1}^{\infty} a_n$ converges, what can you say about the series $\sum_{n=1}^{\infty} \frac{a_n}{a_n+1}$? Same trick. $a_n + 1 \geq 1$, so the terms of $\frac{a_n}{a_n+1} < a_n$, and therefore the series $\sum_{n=1}^{\infty} \frac{a_n}{a_n+1}$ converges.