1. Which of these series converge and which diverge? For each series, write why it either converges or diverges. Use reasons like “It is the tail of a given series” or “it is a geometric series” or “diverges by the no way test”.

(a) \[ \sum_{n=1}^{\infty} \frac{1}{2n^3} \] This series converges as it is a constant times a \( p \)-series having \( p > 1 \).

(b) \[ \sum_{n=1}^{\infty} \frac{n+1}{n} \] The series diverges by the \( n \)-th term test; \( \lim_{n \to \infty} \frac{n+1}{n} = 1 \neq 0 \).

(c) \[ \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2+1}} \] As \( n^2 + 1 > n^2 \), we know that \( \sqrt{n^2 + 1} > \sqrt{n^2} \) and \( n\sqrt{n^2+1} > n\sqrt{n^2} \), making \( \frac{1}{n\sqrt{n^2+1}} < \frac{1}{n\sqrt{n^2}} \). This series is therefore term by term smaller than the \( p \)-series with \( p = 2 \), which converges; hence this series converges.

(d) \[ \sum_{n=1}^{\infty} \frac{6}{4n^2 - 1} \] Rewriting \( \frac{6}{4n^2 - 1} \) as \( \frac{6}{(2n-1)(2n+1)} \), we use partial fractions. The fraction breaks up as \( \sum_{n=1}^{\infty} \frac{3}{2n-1} - \frac{3}{2n+1} \), which is a telescoping series with \( s_n = 3 - \frac{3}{2n+1} \), and thus \( \lim_{n \to \infty} s_n = 3 \).

(e) \[ \sum_{n=1}^{\infty} \left( \frac{e}{\pi} \right)^n \] This is the tail of the geometric series \( \sum_{n=0}^{\infty} \left( \frac{e}{\pi} \right)^n \). As \( \frac{e}{\pi} < 1 \), the series converges.

(f) \[ \sum_{n=1}^{\infty} \frac{\cos^4(\tan^{-1}(n))}{n\sqrt{n}} \] As \( 0 \leq \cos^4(y) \leq 1 \) for ANY value of \( y \), we have that this series is term by term less than or equal to the \( p \)-series having \( p = \frac{5}{4} > 1 \), so the series converges.

(g) \[ \sum_{n=1}^{\infty} e^n n^{-3} \] This series diverges by the \( n \)th term test; \( \lim_{n \to \infty} \frac{e^n}{n^3} = \infty \neq 0 \).

(h) \[ \sum_{n=1}^{\infty} e^{-2n} \] This is a tail of the geometric series \( \sum_{n=0}^{\infty} \left( \frac{1}{e^2} \right)^n \), which has radius \( r = \frac{1}{e^2} < 1 \), so the series converges.
(i) \( \sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} \) If we rewrite the series as \( \sum_{n=0}^{\infty} 5 \left( \frac{-1}{4} \right)^n \), we see that we have a convergent geometric series with \(|r| < 1\).

(j) \( \sum_{n=1}^{\infty} \frac{3}{n+4} \) This series is a constant multiple of a tail of the harmonic series, so it diverges.

(k) \( \sum_{n=0}^{\infty} \frac{1}{n!} \) For \( n \geq 4 \), we know that \( n! \geq 2^n \), so this series is term by term less than or equal to \( \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \) except for a finite number of terms; thus the series converges.

2. Find a closed form (Taylor series) for the following.

(a) \( 1 + \frac{1}{3}x + \frac{1}{3!} \left( \frac{-2}{3} \right)x^2 + \frac{1}{3!} \left( \frac{-2}{3} \right) \left( \frac{-2}{3} \right)x^3 + \ldots \)
   \[ 1 + \sum_{k=1}^{\infty} \left( \frac{1}{3} \right)^k x^k \]

(b) \( 5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \frac{(5x)^7}{7!} + \ldots + \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{(2n+1)!} \)

(c) \( 1 - x^3 + x^6 - x^9 + \ldots + \sum_{n=0}^{\infty} (-x)^{3n} \)

3. Each of the following series is the value of the Taylor series at \( x = 0 \) of a function \( f(x) \) at a particular point. What function and what point? What is the sum of the series?

(a) \( 1 - \frac{2}{3} + \frac{2}{9} - \frac{4}{81} + \ldots + (-1)^n \frac{2^n}{n!3^n} + \ldots e^{-2/3} \)

(b) \( 1 - \frac{\pi^2}{9!} + \frac{\pi^4}{81!4!} - \ldots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \ldots \cos(\pi/3) = \frac{1}{2} \)

4. Here are the first few terms of the Taylor series around 0 of the tangent of \( x \). It converges when \(-\pi/2 < x < \pi/2\):

\[ \tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots \]

Find the first five terms in the series for \( \sec^2(x) \) and for \( \ln |\sec(x)| \). What is the radius of convergence for each of them? Both series have the same radius of convergence as the series for \( \tan x \), which is \( \pi/2 < x < \pi/2 \). The first few terms of the series are:

\[ \sec^2(x) = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots \]

\[ \ln |\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175} + \ldots \]
5. Find the Taylor series for \( \sin^2(x) \) and \( \cos^2(x) \). (Hint: Use double angle formulas) Use the double angle formulas \( \sin^2(x) = \frac{1 - \cos(2x)}{2} \). Write out the Maclaurin series for \( \cos(x) \) and write one for \( \cos(2x) \) using substitution. Then take \( 1 - \cos(2x) \) by canceling out the ones and changing the \((-1)^n\) in the formula for \( \cos(2x) \) to a \((-1)^{n+1}\). Then divide by 2 to get \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!} \). Do something similar for \( \cos^2(x) \); it becomes \( \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!} \).

6. Write the Taylor series for \( \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \).

(a) Use the previous Taylor series to find the Taylor series for \( \tan^{-1}(x) \).

(b) Give an exact value for the following series:

\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \]

\( \tan^{-1}(1) = \pi/4. \)

7. Observe that \( \frac{1}{2+x} = \frac{1}{2(1+(x/2))} = \frac{(1/2)}{1+(x/2)} \). Therefore we may write \( \frac{1}{2+x} = \sum_{n=0}^{\infty} \frac{1}{2} \left( -\frac{x}{2} \right)^n \).

Repeat this process to find a Taylor series for \( \frac{7}{5-x} \):

(a) Centered around \( x = 0 \) \( \frac{7}{5-x} = \frac{7}{1-x/5} = \sum_{n=0}^{\infty} \frac{7}{5} \left( \frac{x}{5} \right)^n \)

(b) Centered about \( x = 2 \) \( \frac{7}{5-x} = \frac{7}{5-x+2-2} = \frac{7}{3-(x-2)-2} = \frac{7}{3-(x-2)} = \frac{(7/3)}{1-\left( \frac{x-2}{3} \right)} \) So we get \( \sum_{n=0}^{\infty} \frac{7}{3} \left( \frac{x-2}{3} \right)^n \)

8. Use the definition of a Taylor series to find the Taylor series for the following functions:

(a) \( \ln(1 + x) \) at \( a = 0. \) \( \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \)

(b) \( \cos(x) \) at \( a = \pi/4. \) NOTE: I meant this problem to say to compute only the first four terms, which are \( \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}(x - \pi/4) - \frac{1}{2\sqrt{2}}(x - \pi/4)^2 + \frac{1}{6\sqrt{2}}(x - \pi/4)^3 + \ldots \)
9. Which of the following statements are true?

(a) If \( a_n \geq 0 \) for every \( n \), then \( \sum_{n=1}^{\infty} a_n \) converges \( \Rightarrow \sum_{n=1}^{\infty} \sqrt{a_n} \) converges. This is false; \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent series, but \( \sum_{n=1}^{\infty} \) is the harmonic series, which clearly diverges.

(b) If \( a_n \geq 0 \) for every \( n \), then \( \sum_{n=1}^{\infty} n a_n \) converges \( \Rightarrow \sum_{n=1}^{\infty} a_n \) converges. This is true; the terms \( n a_n \) are term by term larger than the terms \( a_n \).

(c) If \( a_n \geq 0 \) for every \( n \) and \( a_{n+1} \leq a_n \), and there exists a positive number \( c \) such that \( a_n \geq c \) for every \( n \), then \( \{a_n\} \) converges. This is also true; this is the definition of a monotone bounded series.

10. Find the Maclaurin series for the following functions:

(a) \( \frac{e^x + e^{-x}}{2} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \)

(b) \( \frac{x^2}{1+x} \sum_{n=0}^{\infty} x^{n+2} \)

(c) \( x \cos(\pi x) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!} \)

(d) \( e^x - (1 + x) \sum_{n=2}^{\infty} \frac{x^n}{n!} \)

11. What happens if you add a finite number of terms to a convergent series? A divergent series? What happens if you multiply a convergent series by a nonzero constant? A divergent series?

None of these operations alter the convergence or divergence of a series. This is the concept behind the tail of a series.

12. If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are both convergent series of nonnegative numbers, what can be said about \( \sum_{n=1}^{\infty} a_n b_n \)? As \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are convergent, we know the terms are approaching zero, and thus at some point become smaller than one. Therefore \( a_n b_n \leq a_n \), and the series \( \sum_{n=1}^{\infty} a_n b_n \) converges.
13. If $a_n \geq 0$ for every $n$ and $\sum_{n=1}^{\infty} a_n$ converges, what can you say about the series
\[ \sum_{n=1}^{\infty} \frac{a_n}{a_n + 1} \]?
Same trick. $a_n + 1 \geq 1$, so the terms of $\frac{a_n}{a_n + 1} < a_n$, and therefore the series $\sum_{n=1}^{\infty} \frac{a_n}{a_n + 1}$ converges.