1 Integrate. (a) \( \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} \, dx \) (b) \( \int \frac{x \, dx}{x^2 + 2x + 2} \) (c) \( \int \frac{dx}{(x-1)(x+1)(x+2)} \) (d) \( \int \frac{dx}{\cos^4(x)} \) (e) \( \int \frac{(3x+1) \, dx}{\sqrt{3x^2 + 2x + 1}} \)

(a) \( \frac{2\pi - 3\sqrt{3}}{24} \) (Trig substitution, \( x = \sin(\theta) \))

(b) \( \frac{1}{2} \ln |x^2 + 2x + 2| - \tan^{-1}(x + 1) + c \) (Complete the square, then \( u = x + 1 \).)

(c) \( \frac{1}{6} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + \frac{1}{3} \ln |x + 3| + c \) (Partial Fractions)

(d) \( \frac{1}{3} \tan^3(x) + \tan(x) + c \) (Change to \( \sec^2(x) \sec^2(x) = (\tan^2(x) + 1) \sec^2(x) \), then use \( u \)-substitution.)

(e) \( (3x^2 + 2x + 1)^{\frac{1}{2}} + c \) (Try \( u = 3x^2 + 2x + 1 \))

2 For each of the following, either compute the value or write DIVERGES.

(a) \( \int_1^\infty \frac{dx}{x^2} \) (b) \( \int_0^1 \frac{dx}{\sqrt{x}} \) (c) \( \int_0^\infty \sin(x) \, dx \) (d) \( \int_{-1}^1 \frac{dx}{x^3} \)

(a) 1

(b) 2

(c) Diverges (Note that \( \int_0^n \sin(x) \, dx = 2 \) if \( n \) is odd, and 0 if \( n \) is even.)

(d) Diverges (Inconclusive result \( \infty - \infty \))
3  Integrate.  (a) \( \int x^2 e^{x^3} \, dx \)  (b) \( \int \sin^2(2x) \, dx \)  (c) \( \int x^4 \ln(x) \, dx \)  (d) \( \int \sin^3(x) \, dx \)

(a) \( \frac{1}{3}x^3 + x \) (Try \( u = x^3 \))

(b) \( \frac{x}{2} - \frac{\sin(4x)}{8} + c \) (Try power-reducing formula \( \sin^2(2x) = \frac{1 - \cos(4x)}{2} \))

(c) \( \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + c \) (Try integration by parts, \( u = \ln(x) \), \( dv = x^4 \, dx \))

(d) \( \frac{1}{3} \cos^3(x) - \cos(x) + c \) (Try rewriting as \( (1 - \cos^2(x)) \sin(x) \), then let \( u = \cos(x) \))

4  Solve these initial value problems.

(a) \( \frac{dy}{dx} - \frac{y}{2x} = x \), \( y(3) = 9 \)  (b) \( \frac{d^2 y}{dx^2} = \frac{dy}{dx} + e^x \), \( y(0) = 1 = y'(0) \).

(a) \( y = \frac{2}{3}x^2 + \sqrt{3}x \) (Use integrating factor \( \mu(x) = \frac{1}{x^2} \))

(b) \( y = xe^x + 1 \) (Use integrating factor \( \mu(x) = e^{-x} \))

5  Find the general solution for \( y'' - 4y' + 13y = 0 \).

\( y = e^{2t} \left( c_1 \cos(3t) + c_2 \sin(3t) \right) \) (Characteristic values are \( 2 \pm 3i \))

6  Write the general solution for each of these.

(a) \( y'' + 6y' + 9y = e^x \)  (b) \( y'' + 4y = x^2 \).

(a) \( y = c_1 e^{-3x} + c_2 xe^{-3x} + \frac{1}{16}e^x \) (To find the kernel, note that the equation has repeated real eigenvalues \(-3, -3\). To find particular solution, suppose \( y = Ae^x \).)

(b) \( y = c_1 \sin(2x) + c_2 \cos(2x) + \frac{x^2}{4} - \frac{1}{8} \) (To find kernel, note that the equation has eigenvalues \( \pm 2i \). To find particular solution, suppose \( y = Ax^2 + Bx + C \).)

7  Use Simpson’s rule with four intervals to find an approximation for \( \ln(3) \). HINT: Start by writing a simple integral whose value is \( \ln(3) \). Work with fractions, not decimals.

NOTE: \( \ln(3) \approx 1.098612289 \). You can use this to check that your answer is reasonable.

Should get \( \ln(3) \approx 1.1 \)
I am trying to evaluate \( \int_{-\sqrt{2}}^{-1} \frac{x^2 \, dx}{\sqrt{2x^2 - 1}} \). I decide to try a trig substitution, and I obtain a definite integral involving trig functions. What integral do I get? Do not evaluate.

In order to make both the trigonometric substitution easier and the bounds nicer, we first make the following \( u \)-substitution:

\[
u = -\sqrt{2}x
\]

\[
du = -\sqrt{2} \, dx
\]

This substitution implies that \( x = -\frac{1}{\sqrt{2}} u \), and our bounds change from \(-\sqrt{2} \) and \(-1 \) to \(2 \) and \( \sqrt{2} \), respectively. This therefore changes the integral as shown below:

\[
\int_{-\sqrt{2}}^{-1} \frac{x^2 \, dx}{\sqrt{2x^2 - 1}} = -\frac{1}{2\sqrt{2}} \int_{2}^{\sqrt{2}} \frac{u^2 \, du}{\sqrt{u^2 - 1}} = \frac{1}{2\sqrt{2}} \int_{2}^{\sqrt{2}} \frac{u^2 \, du}{\sqrt{u^2 - 1}}
\]

We now make the trigonometric substitution \( u = \sec \theta \) so that \( du = \sec \theta \tan \theta \, d\theta \). Remembering to change our bounds and using the Pythagorean identity \( 1 + \tan^2 \theta = \sec^2 \theta \), we have:

\[
\frac{1}{2\sqrt{2}} \int_{2}^{\sqrt{2}} \frac{u^2 \, du}{\sqrt{u^2 - 1}} = \frac{1}{2\sqrt{2}} \int_{\pi/4}^{\pi/3} \sec^3 \theta \tan \theta \, d\theta \]

It remains to be determined if we should replace \( \sqrt{\tan^2 \theta} \) by \( \tan \theta \) or by \(- \tan \theta \). We realize that we can replace \( \sqrt{\tan^2 \theta} = \tan \theta \) since \( \tan \theta > 0 \) on the interval we are concerned with, the interval from \( \pi/4 \) to \( \pi/3 \). Thus the final integral obtained is:

\[
\frac{1}{2\sqrt{2}} \int_{\pi/4}^{\pi/3} \sec^3 \theta \, d\theta
\]

Integrate.

(a) \( \int \frac{\sin(x) \, dx}{\sqrt{1 + \cos(x)}} \) 
(b) \( \int \left( \frac{1 + x}{x} \right)^2 \, dx \) 
(c) \( \int \tan^3(x) \sec(x) \, dx \) 
(d) \( \int_1^c \frac{dx}{x + x \ln(x)} \)

(a) Let \( u = 1 + \cos x \). Then \( du = -\sin(x) \, dx \), and:

\[
\int \frac{\sin(x) \, dx}{\sqrt{1 - \cos(x)}} = -\int \frac{1}{\sqrt{u}} \, du = -2u^{1/2} + c = -2\sqrt{1 + \cos(x)} + c
\]

(b) Distributing the square to both the numerator and denominator, we get \( \int \left( \frac{1 + x}{x} \right)^2 \, dx = \int \frac{1 + 2x + x^2}{x^2} \, dx \). Now if we break up the fraction, we get:

\[
\int \frac{1 + 2x + x^2}{x^2} \, dx = \int \left( x^{-2} + \frac{2}{x} + 1 \right) \, dx = -\frac{1}{x} + 2 \ln |x| + x + c
\]
(c) Rewrite \( \tan^3(x) \) as \((\tan^2(x)) \tan(x) \sec(x)\), and use the Pythagorean identity \(1 + \tan^2(\theta) = \sec^2(\theta)\) to write \(\tan^2(x) = \sec^2(x) - 1\). So \(\int \tan^3(x) \sec(x) \, dx = \int (\sec^2(x) - 1) \sec(x) \tan(x) \, dx\), and we can do a \(u\)-sub. Then \(u = \sec(x)\) and \(du = \sec(x) \tan(x) \, dx\), making the integral:

\[
\int (1 - u^2) \, du = u - \frac{1}{3} u^3 + c = \sec(x) + \frac{1}{3} \sec^3(x) + c
\]

(d) We factor the denominator as \(x(1 + \ln(x))\), and then do a \(u\)-substitution, where \(u = 1 + \ln(x)\) and \(du = \frac{1}{x} \, dx\). We therefore change the bounds on the integral to be from 1 to 2. So:

\[
\int_1^e \frac{dx}{x + x \ln(x)} = \left[ \frac{1}{u} \right]_1^2 = \ln(2) - \ln(1) = \ln(2)
\]

10 Integrate. (a) \(\int_0^\pi/9 \tan(3x) \, dx\)  (b) \(\int \frac{dt}{4t^3 - t}\)  (c) \(\int_2^\sqrt{3} \frac{x^2 - 3}{x} \, dx\)  (d) \(\int_4^1 e^{\sqrt{x}} \, dx\)

(a) We use the \(u\)-sub \(u = 3x\) and then \(du = 3 \, dx\). So:

\[
\int_0^{\pi/9} \tan(3x) \, dx = \frac{1}{3} \int_0^{\pi/3} \tan(u) \, du = \frac{1}{3} \ln(\sec(u)) \bigg|_0^{\pi/3} = \frac{1}{3} \ln(2)
\]

(b) We factor the denominator of the fraction and write \(4t^3 - t = t(2t - 1)(2t + 1)\). Then use partial fractions and the cover up method to get:

\[
\int \frac{dt}{4t^3 - t} = \int \left( \frac{-1}{t} + \frac{1}{2t + 1} + \frac{1}{2t - 1} \right) \, dt
\]

\[
= - \ln |t| + \frac{1}{2} \ln |2t + 1| + \frac{1}{2} \ln |2t - 1| + c
\]

(c) We first do a \(u\)-substitution, letting \(u = \frac{1}{\sqrt{3}}x\), and then \(du = \frac{1}{\sqrt{3}} \, dx\). This allows us to change the bounds on the integral from \(\sqrt{3}\) and 2 to the bounds going from 1 to \(\frac{2}{\sqrt{3}}\), and to change the given integral into \(\sqrt{3} \int_1^{2/\sqrt{3}} \frac{\sqrt{3}(u^2 - 1)}{\sqrt{3}u} \, du = \sqrt{3} \int_1^{2/\sqrt{3}} \frac{u^2 - 1}{u} \, du\). We can now to a trigonometric substitution, letting \(u = \sec(\theta)\), and hence \(du = \sec(\theta) \tan(\theta) \, d\theta\). This gives us the new integral:

\[
\sqrt{3} \int_0^{\pi/6} \frac{\sqrt{\tan^2(\theta)}}{\sec(\theta)} \sec(\theta) \tan(\theta) \, d\theta
\]
As \( \tan(\theta) \) is positive on the range from 0 to \( \pi/6 \), we can further simplify the integral to obtain \( \sqrt{3} \int_0^{\pi/6} \tan^2(\theta) d\theta \), and we can use the Pythagorean identity to finally obtain \( \sqrt{3} \int_0^{\pi/6} (\sec^2(\theta) - 1) d\theta \). We can now integrate this, giving \( \sqrt{3} (\tan(\theta) - \theta) \bigg|_0^{\pi/6} \).

If we evaluate, we get a final answer of \( 1 - \sqrt{3} \pi/6 \).

(d) We make the \( y \)-substitution \( y = \sqrt{x} \). Then \( dy = \frac{1}{2\sqrt{x}} dx \), so we can write \( dx = 2y dy \). Subbing this in for the integral and changing our bounds, we have:

\[
\int_1^4 e^{\sqrt{x}} dx = 2 \int_1^2 ye^{y} dy
\]

We can now perform integration by parts on this new integral to find the solution:

\[
2 \int_1^2 ye^{y} dy = 2(ye^{y} - e^{y}) \bigg|_1^2 = 2(2e^2 - e^2) - (e - e) = 2e^2
\]

\[\blacksquare\]

11 For each of the following integrals, either compute the value, or write DIVERGENT.

| (a) \( \int_{-1}^{2} \frac{dx}{x^3} \) | (b) \( \int_{0}^{\infty} xe^{-x} \) | (c) \( \int_{-1}^{2} \frac{dx}{e^x} \) | (d) \( \int_{0}^{\infty} \frac{x \, dx}{x + 1} \) |

(a) Officially, we know that the given integral can be represented as follows:

\[
\lim_{b \to 0^-} \int_{-1}^{b} x^{-3} dx + \lim_{a \to 0^+} \int_{a}^{2} x^{-3} dx
\]

However, we will show that the first integral diverges, and therefore the entire integral diverges by the “no way” test. So:

\[
\lim_{b \to 0^-} \int_{-1}^{b} x^{-3} dx = \lim_{b \to 0^-} \frac{-1}{x^2} \bigg|_{-1}^{b} = \lim_{b \to 0^-} \left( -1 + \frac{1}{b^2} \right) = \infty
\]

Therefore the integral is divergent.

(b) We rewrite the improper integral as follows:

\[
\lim_{b \to \infty} \int_{0}^{b} xe^{-x} \, dx
\]
We now compute the integral using integration by parts, and evaluate the limit.

\[
\lim_{b \to \infty} \int_0^b xe^{-x} \, dx
\]

\[
= \lim_{b \to \infty} \left[ -xe^{-x} \right]_0^b + \int_0^b e^{-x} \, dx
\]

\[
= \lim_{b \to \infty} \left( -xe^{-x} - e^{-x} \right) - \int_0^b e^{-x} \, dx
\]

\[
= \lim_{b \to \infty} \left( -b - e^{-x} \right) = 1
\]

(c) \( \int_{-1}^2 \frac{dx}{e^x} = -e^{-x} \bigg|_{-1}^{2} = -\frac{1}{e^2} + e. \)

(d) We turn the improper integral into a proper integral and then take a limit:

\[
\lim_{b \to \infty} \int_0^b \frac{x}{x + 1} \, dx
\]

Now that the integral makes sense, we do polynomial long division and see that we can break up the quantity \( \frac{x}{x+1} = 1 - \frac{1}{x+1}. \) We then evaluate:

\[
\lim_{b \to \infty} \int_0^b \left( 1 - \frac{1}{x+1} \right) \, dx
\]

\[
= \lim_{b \to \infty} \left( x - \ln |1 + x| \right) - \left. \right|_0^b
\]

\[
= \lim_{b \to \infty} \left( b - \ln |1 + b| \right)
\]

If we rewrite \( b \) as \( \ln(e^b) \), we can combine these two quantities together, and we then must compute \( \lim_{b \to \infty} \frac{e^b}{b + 1} = \lim_{b \to \infty} \ln(b) = \infty, \) where this last equality holds because \( e^x \) grows much faster than any polynomial, and therefore \( \lim_{b \to \infty} \frac{e^b}{1 + b} = \infty. \) Hence the integral diverges.
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(a) \( \int_0^\pi \sin^2(x/3) \, dx = \frac{1}{2} \int_0^\pi 1 - \cos(2x/3) \, dx = \frac{1}{2} \left( x - \frac{3}{2} \sin(2x/3) \right) \bigg|_0^\pi = \frac{\pi}{2} - \frac{3\sqrt{3}}{8} \)

(b) We first perform polynomial long division, rewriting the integral as \( \int x + \frac{-5x}{x^2 + 5} \, dx = \int x \, dx - \int \frac{5x}{x^2 + 5} \, dx \). On the second integral, we make the \( u \)-substitution \( u = x^2 + 5 \), and then \( du = 2x \, dx \), so the integral is equal to \( \frac{1}{2}x^2 - \frac{5}{2} \ln |x^2 + 5| + c \).

(c) Note that the denominator factors as \((x + 3)(x + 5)\). We can therefore use partial fractions and the cover up method to break up the integral as \( \int \frac{3}{x + 3} - \frac{2}{x + 2} \, dx \), which we integrate to get \( 3 \ln |x + 3| = 2 \ln |x + 2| + c \).

(d) Let \( y = \sqrt{x} \). Then \( dy = \frac{1}{\sqrt{x}} \, dx \), or equivalently, \( dx = 2 \sqrt{y} \, dy \). We then have:

\[
\int_1^2 \ln(\sqrt{x}) \, dx = 2 \int_1^2 y \ln(y) \, dy
\]

This new integral can be done using integration by parts, where we set \( u = \ln(y) \) and \( dv = y \, dy \). We therefore get that this integral is equal to \( 2(\frac{1}{2}y^2 \ln(y) - \frac{1}{4}y^2) \bigg|_1^e = \frac{1}{2}(e^2 + 1) \).

(e) Let \( u = 3 - 2 \cos(x) \). Then \( du = 2 \sin(x) \, dx \) and we have:

\[
\int_0^{\pi/2} \frac{\sin(x) \, dx}{(3 - 2 \cos(x))^2} = \frac{1}{2} \int_1^3 \frac{du}{u^2}
\]

We can now integrate to see that this is equal to \( \frac{1}{2u} \bigg|_1^3 = \frac{1}{3} \).
Integrate. 

(a) \[ \int \frac{x^5 \, dx}{x^3 + 1} \]

(b) \[ \int x^2 e^x \, dx \]

(c) \[ \int \left( \frac{dx}{(4 + 5x^2)^{3/2}} \right) \]

(d) \[ \int \sec^6(x) \, dx \]

(a) Do polynomial long division to rewrite \[ \int \frac{x^5 \, dx}{x^3 + 1} = \int x^2 \, dx - \int \frac{x^2}{x^3 + 1} \, dx \], where we can integrate the second term using a \( u \)-substitution where \( u = x^3 + 1 \) and therefore \( du = 3x^2 \, dx \). Hence the integral is \( \frac{1}{3} x^3 - \frac{1}{3} \ln |x^3 + 1| + c \).

(b) We use integration by parts twice on this problem. We first let \( u = x^2 \), so \( du = 2x \, dx \) and set \( dv = e^{-x} \, dx \) so that \( v = -e^{-x} \). Then:

\[ \int x^2 e^{-x} \, dx = -x^2 e^{-x} + 2 \int xe^{-x} \, dx \]

If we set \( J = \int xe^{-x} \, dx \), then we now use integration by parts again on \( J \) with \( u = x \), \( du = dx \), \( dv = e^{-x} \, dx \) and \( v = -e^{-x} \). Then we have that \( J = -xe^{-x} + \int e^{-x} \, dx \), so \( J = -xe^{-x} - e^{-x} + c \). So altogether, the integral is equal to \( -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c \).

(c) First, factor out a 5 from the denominator; that is, write \( (4 + 5x^2)^{3/2} = (5\left( \frac{4}{5} + x^2 \right))^{3/2} \). It will be useful to think of \( (5\left( \frac{4}{5} + x^2 \right))^{3/2} \) as \( (\sqrt{5\left( \frac{4}{5} + x^2 \right)})^3 \). We now do a trig substitution with \( x = \frac{2}{\sqrt{5}} \tan(\theta) \), so that \( dx = \frac{2}{\sqrt{5}} \sec^2(\theta) \, d\theta \). We then use the identity \( 1 + \tan^2(\theta) = \sec^2(\theta) \) to rewrite this integral as:

\[ \frac{2}{\sqrt{5}} \int \frac{\sec^2(\theta) \, d\theta}{\left( \frac{4}{\sqrt{5}} \sec^2(\theta) \, d\theta \right)^{3/2}} = \frac{2}{\sqrt{5}} \int \frac{\sec^2(\theta)}{8\sec^3(\theta)} = \frac{1}{4\sqrt{5}} \int \cos(\theta) \, d\theta \]

Hence the integral evaluates to \( \frac{1}{4\sqrt{5}} \sin(\theta) + c \). If we draw in our reference triangle for the original substitution to substitute back in for \( x \), we see that the solution is \( \frac{x}{\sqrt{4x^2 + 5}} + c \).

(d) We replace \( \sec^4(x) \) by \( (1 + \tan^2(x))^2 \), and then do a \( u \)-substitution where \( u = \tan(x) \) and then \( du = \sec^2(x) \, dx \), and then \( \int \sec^6(x) \, dx = \int (1 - u^2)^2 \, du = \int 1 - 2u^2 + u^4 \, du \). We can then integrate this, and find that the solution is \( \tan(x) - \frac{2}{3} \tan^3(x) + \frac{1}{5} \tan^5(x) + c \).
14. Decide whether or not this integral has a meaningful value. If it does, compute the value and if not, explain why not: \( \int_{-1}^{2} \frac{dx}{\sqrt{|x|}} \).

\[
\int_{-1}^{2} \frac{dx}{\sqrt{|x|}} = \int_{-1}^{0} \frac{dx}{\sqrt{|x|}} + \int_{0}^{2} \frac{dx}{\sqrt{|x|}}
\]

Since \(|x| = -x\), if \(x \leq 0\) and \(|x| = x\), if \(x \geq 0\), have

\[
\int_{-1}^{2} \frac{dx}{\sqrt{|x|}} = \int_{-1}^{0} \frac{dx}{\sqrt{-x}} + \int_{0}^{2} \frac{dx}{\sqrt{x}}
\]

\[
\int_{0}^{-1} \frac{dx}{\sqrt{-x}} + \int_{0}^{2} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^-} [-2\sqrt{-a}] + \lim_{b \to 0^+} [2\sqrt{x}] = -2 + 2 = 0
\]

\[
\int_{0}^{-1} \frac{2}{\sqrt{-x}} + \int_{0}^{2} \frac{2}{\sqrt{x}} = 2 + 2\sqrt{2}
\]

15. Integrate

\[
\begin{align*}
(a) & \int \sec^4(x) \, dx \\
(b) & \int \frac{x^7 + x^3}{x^4 - 1} \, dx \\
(c) & \int \frac{(2x + 3)}{4x^2 + 4x + 5} \, dx \\
(d) & \int_0^2 (4 - x^2)^{3/2} \, dx \\
(e) & \int \frac{\cos(x)}{\sin^2(x) - 3 \sin(x) + 2} \, dx \\
(f) & \int x \sin(\ln(x)) \, dx
\end{align*}
\]

(a) \( \int \sec^4(x) \, dx = \int \sec^2(x) \sec^2(x) \, dx = \int (1 + \tan^2(x)) \sec^2(x) \, dx \)

Let \( u = \tan^2(x), du = \sec^2(x) \, dx \)

\[
I = \int (1 + u^2) \, du = u + \frac{u^3}{3} + C = \tan(x) + \frac{\tan^3(x)}{3} + C
\]

(b) \( \int \frac{x^7 + x^3}{x^4 - 1} \, dx \)

Use long division, get \( I = \int (x^3 + \frac{2x^3}{x^4 - 1}) \, dx = \frac{x^4}{4} + \int \frac{2x^3}{x^4 - 1} \, dx = \frac{x^4}{4} + I_1 \)

Let \( u = x^2, du = 2x \, dx \)

\[
I_1 = \int \frac{udu}{u^2 - 1}
\]

\[
\frac{u}{u^2 - 1} = \frac{u}{(u - 1)(u + 1)} = \frac{A}{u - 1} + \frac{B}{u + 1}
\]

\[
A = \frac{1}{2}, B = \frac{1}{2}
\]

\[
I_1 = \frac{1}{2} \ln |u - 1| + \frac{1}{2} \ln |u + 1| + C = \frac{1}{2} \ln |x^2 - 1| + \frac{1}{2} \ln |x^2 + 1| + C
\]

\[
I = \frac{x^4}{4} + \frac{1}{2} \ln |x^2 - 1| + \frac{1}{2} \ln |x^2 + 1| + C
\]
(c) \[
\int \frac{2x + 3}{4x^2 + 4x + 5} \, dx = \int \frac{(2x + 1) + 2}{(4x^2 + 4x + 1) + 4} \, dx = \int \frac{(2x + 1) + 2}{(2x + 1)^2 + 4} \, dx
\]
Let \( u = 2x + 1, du = 2dx \)

\[
I = \frac{1}{2} \int \frac{u + 2}{u^2 + 4} \, du = \frac{1}{2} \int \frac{u}{u^2 + 4} \, du + \frac{1}{2} \int \frac{1}{u^2 + 4} \, du = \frac{1}{2} I_1 + I_2
\]
Let \( v = u^2 + 4, dv = 2udu \)

\[
I_1 = \frac{1}{2} \int \frac{dv}{v} = \frac{1}{2} \ln |v| + C = \frac{1}{2} \ln (u^2 + 4) + C
\]
Let \( u = 2w, du = 2dw, u^2 = 4w^2 \)

\[
I_1 = \int \frac{2dw}{4(w^2 + 1)} = \frac{1}{2} \int \frac{dw}{w^2 + 1} = \frac{1}{2} \arctan(w) + C = \frac{1}{2} \arctan \left( \frac{u}{2} \right) + C
\]

\[
I = \frac{1}{4} \ln (u^2 + 4) + \frac{1}{2} \arctan \left( \frac{u}{2} \right) + C
\]
\[
= \frac{1}{4} \ln ((2x + 1)^2 + 4) + \frac{1}{2} \arctan \left( \frac{2x + 1}{2} \right) + C
\]
\[
= \frac{1}{4} \ln (4x^2 + 4x + 5) + \frac{1}{2} \arctan \left( \frac{2x + 1}{2} \right) + C
\]

(d) \[
\int_0^2 (4 - x^2)^{\frac{3}{2}} \, dx
\]
Let \( x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \theta = \arcsin \frac{x}{2}. \) If \( x = 0, \) then \( \theta = 0. \) If \( x = 2, \) then \( \theta = \frac{\pi}{2}. \) Double angle formula: \( \cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta. \)
\[ I = \int_0^{\frac{\pi}{2}} (4 - 4 \sin^2 \theta)^{\frac{3}{2}} 2 \cos \theta d\theta \\
= \int_0^{\frac{\pi}{2}} 8 \cos^3 \theta 2 \cos \theta d\theta \\
= 16 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta \\
= 16 \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta \\
= 4 \int_0^{\frac{\pi}{2}} (1 + 2 \cos 2\theta + \cos^2 \theta) d\theta \\
= 4[\theta + \sin 2\theta]_0^{\frac{\pi}{2}} + 4 \int_0^{\frac{\pi}{2}} \cos^2 2\theta d\theta \\
= 4\left(\frac{\pi}{2} + \sin \pi - 0 - \sin 0\right) + 4 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 4\theta}{2} d\theta \\
= 2\pi + 2 \int_0^{\frac{\pi}{2}} (1 + \cos 4\theta) d\theta \\
= 2\pi + 2[\theta + \frac{1}{4} \sin 4\theta]_0^{\frac{\pi}{2}} \\
= 2\pi + 2\left(\frac{\pi}{2} + \frac{1}{4} \sin 2\pi - 0 - \frac{1}{4} \sin 0\right) \\
= 2\pi + \pi = 3\pi \\
\]

(e) \[ \int \frac{\cos x dx}{\sin^2 x - 3 \sin x + 2} \]
Let \( u = \sin x \)
\( du = \cos x dx. \)
\[ \int \frac{du}{u^2 - 3u + 2} = \int \frac{du}{(u - 2)(u - 1)} \]
\[ \frac{du}{(u - 2)(u - 1)} = \frac{A}{u - 2} + \frac{B}{u - 1} \]
\[ A = 1B = -1 \]
\[ I = \int \left(\frac{1}{u - 2} - \frac{1}{u - 1}\right) du \\
= \ln |u - 2| - \ln |u - 1| + C = \ln |\sin x - 2| - \ln |\sin x - 1| + C \]
(f) \( \int x \sin(\ln(x)) \, dx \)

Use integration by parts, \( u = \sin(\ln(x)) \), \( dv = x \, dx \), \( du = \cos(\ln(x)) \frac{1}{x} \), \( v = \frac{x^2}{2} \).

\[
I = \frac{x^2}{2} \sin(\ln(x)) - \int \frac{x^2}{2} \cos(\ln(x)) \, dx = \frac{x^2}{2} \sin(\ln(x)) - \frac{1}{2} \int x \cos(\ln(x)) \, dx = \frac{x^2}{2} \sin(\ln(x)) - \frac{1}{2} I
\]

Use integration by parts again, \( u = \cos(\ln(x)) \), \( dv = x \, dx \), \( du = -\sin(\ln(x)) \frac{1}{x} \), \( v = \frac{x^2}{2} \).

\[
I = \frac{x^2}{2} \cos(\ln(x)) + \int \frac{x^2}{2} \sin(\ln(x)) \, dx = \frac{x^2}{2} \cos(\ln(x)) + \frac{1}{2} \int x \sin(\ln(x)) \, dx = \frac{x^2}{2} \cos(\ln(x)) + \frac{1}{2} I
\]

\[
I = \frac{x^2}{2} \sin(\ln(x)) - \frac{x^2}{4} \cos(\ln(x)) - \frac{1}{4} I
\]

\[
\frac{5}{4} I = \frac{x^2}{2} \sin(\ln(x)) - \frac{x^2}{4} \cos(\ln(x))
\]

\[
I = \frac{2}{5} x^2 \sin(\ln(x)) - 5x^2 \cos(\ln(x)) + C
\]

16 State which of the following are DIVERGENT improper integrals.

(a) \( \int_{0}^{\infty} \cos(x) \, dx \)  (b) \( \int_{0}^{\infty} \frac{dx}{2 + x^2} \)  (c) \( \int_{-1}^{1} \frac{dx}{\sqrt{x}} \)  (d) \( \int_{0}^{\pi} \tan(x) \, dx \)  (e) \( \int_{0}^{1} \frac{x \, dx}{x^2 - 1} \)

(a) By the graph of \( \cos x \), we know that this integral diverges.

(b) \( \int_{0}^{\infty} \frac{dx}{2 + x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{2 + x^2} \)

Let \( x = \sqrt{2}u \), \( dx = \sqrt{2} \, du \)

\[
\int \frac{dx}{2 + x^2} = \int \frac{\sqrt{2} \, du}{2 + 2u^2} = \frac{1}{\sqrt{2}} \int \frac{du}{1 + u^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C
\]

\[
I = \lim_{b \to \infty} \left[ \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_{0}^{b} = \frac{1}{\sqrt{2}} \lim_{b \to \infty} \left( \tan^{-1}\left(\frac{b}{\sqrt{2}}\right) - \tan^{-1}(0) \right) = \frac{1}{\sqrt{2}} \left( \frac{\pi}{2} - 0 \right) = \frac{\sqrt{2}\pi}{4}
\]

(c) \( \int_{-1}^{1} \frac{dx}{\sqrt{x}} = \int_{-1}^{0} \frac{dx}{\sqrt{x}} + \int_{0}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0} \int_{-1}^{a} \frac{dx}{\sqrt{x}} + \lim_{b \to 0} \int_{b}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0} \left[ 3x^{\frac{3}{2}} \right]_{-1}^{a} + \lim_{b \to 0} \left[ 3x^{\frac{3}{2}} \right]_{1}^{b} \)

So, \( \int_{-1}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0} \left( \frac{3}{2} a^\frac{3}{2} - \frac{3}{2} \right) + \lim_{b \to 0^+} \left( \frac{3}{2} b^\frac{3}{2} - \frac{3}{2} b^\frac{3}{2} \right) = -\frac{3}{2} + \frac{3}{2} = 0 \)
(d) \[ \int_0^\pi \tan x \, dx = \int_0^{\pi/2} \tan x \, dx + \int_{\pi/2}^\pi \tan x \, dx = \lim_{a \to \pi/2^-} \frac{\ln |\sec x|}{|a|} + \lim_{b \to \pi^+} \frac{\ln |\sec x|}{b} \]

So, \[ \int_0^\pi \tan x \, dx = \lim_{a \to \pi/2^-} (\ln |\sec a| - \ln |\sec 0|) + \lim_{b \to \pi^+} (\ln |\sec \pi| - \ln |\sec b|) . \]

But, \[ \lim_{a \to \pi/2^-} \ln |\sec a| \text{ diverges.} \]

Thus, the integral \( \int_0^\pi \tan x \, dx \) diverges.

(e) \[ \int_0^1 \frac{xdx}{x^2 - 1} = \lim_{a \to 1^-} \int_0^a \frac{xdx}{x^2 - 1} \]

Let \( u = x^2 - 1, du = 2xdx \).

\[ \int \frac{xdx}{x^2 - 1} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u + C = \frac{1}{2} \ln |x^2 - 1| + C \]

\[ I = \lim_{a \to 1^-} \left[ \frac{1}{2} \ln |x^2 - 1| \right]_0^a = \lim_{a \to 1^-} \left[ \frac{1}{2} \ln |a^2 - 1| - \frac{1}{2} \ln 1 \right] = \frac{1}{2} \lim_{a \to 1^-} \ln |a^2 - 1| = -\infty \]

So the integral diverges.

---

17. Integrate. (a) \[ \int_0^1 \frac{x + 1}{x^2 + 2x - 4} \, dx \] (b) \[ \int x \sec^2(x) \, dx \] (c) \[ \int \cos^4(x/4) \, dx \]

(a) \[ \int_0^1 \frac{x + 1}{x^2 + 2x - 4} \, dx \]

Let \( u = x^2 + 2x - 4, du = (2x + 2)dx = 2(x + 1)dx, x = 1 \Rightarrow u = -1; x = 0 \Rightarrow u = -4 \).

\[ I = \frac{1}{2} \int_{-4}^{-1} \frac{du}{u} = \frac{1}{2} \ln |u| \bigg|_{-4} = \frac{1}{2} (\ln 1 - \ln 4) = -\frac{1}{2} \ln 4 = -\ln 2 \]

(b) \[ \int x \sec^2 x \, dx \]

Let \( u = x, dv = \sec^2 x \, dx; du = dx, v = \int \sec^2 x \, dx = \tan x \).

\[ I = x \tan x - \int \tan x \, dx = x \tan x - \ln |\sec x| + C \]

(c) \[ \int \cos^4(x/4) \, dx = \int \left( \frac{1 + \cos(x/2)}{2} \right)^2 \, dx = \frac{1}{4} \int (1 + 2 \cos(x/2) + \cos^2(x/2)) \, dx = \frac{1}{4} \int (x + 4 \sin(x/2) + \int \cos^2(x/2) \, dx) = \frac{1}{4} \int \left( \frac{1 + \cos x}{2} \right) \, dx = \frac{1}{4} \left( x + 4 \sin(x/2) + \frac{x}{2} \sin x \right) + C = \frac{3}{8} x + \sin(x/2) + \frac{1}{8} \sin x + C \]

\[ \blacksquare \]
18  Integrate.  (a) \( \int_0^\pi \sin^2(x/3) \, dx \).  (b) \( \int \frac{x^3 \, dx}{x^2 + 5} \).  (c) \( \int \frac{x \, dx}{x^2 + 5x + 6} \).

(a) \[ \int_0^\pi \sin^2(x/3) \, dx = \int_0^\pi \frac{1 - \cos(2x/3)}{2} \, dx = \left[ \frac{x}{2} \right]_0^\pi - \frac{1}{2} \int_0^\pi \cos(2x/3) \, dx = \frac{1}{2}(\pi - 0) - \frac{1}{2} \left[ \frac{3}{2} \sin(2x/3) \right]_0^\pi = \frac{\pi}{2} - \frac{3}{4}(\sin(\frac{2\pi}{3}) - \sin(0)) = \frac{\pi}{2} - \frac{3}{4}(\sqrt{3} - 0) = \frac{\pi}{2} - \frac{3\sqrt{3}}{8} \]

(b) \[ \int \frac{x^3 \, dx}{x^2 + 5} \]

Let \( u = x^2 + 5, x^2 = u - 5, du = 2x \, dx \).

\[ I = \int \frac{x^2 \, dx}{u} = \frac{1}{2} \int \frac{u - 5}{u} \, du = \frac{1}{2} \int (1 - \frac{5}{u}) \, du = \frac{1}{2} u - \frac{5}{2} \ln |u| + C = \frac{1}{2}(x^2 + 5) - \frac{5}{2} \ln(x^2 + 5) + C = \frac{1}{2}x^2 - \frac{5}{2} \ln(x^2 + 5) + C \]

(c) \[ \int \frac{x \, dx}{x^2 + 5x + 6} \]

\[ \frac{x}{(x + 3)(x + 2)} = \frac{A}{x + 3} + \frac{B}{x + 2} \]

\( A = 3, B = -2 \)

\[ I = 3 \ln |x + 3| - 2 \ln |x + 2| + C \]

19  Find general solutions.

(a) \( y'' + y' - 2y = 4x \)  (b) \( y''' - 2y'' + y' = 1 \)  (c) \( y'' - 2y' + 5y = 0 \).

(a) Let’s solve the complementary equation \( y'' + y' - 2y = 0 \), substitution of \( y = e^{rx} \) into the differential equation yields the auxiliary equation. \( r^2 + r - 2 = (r + 2)(r - 1) = 0 \)

So \( r = -2 \) and \( r = 1 \) give the complementary solution \( y_c = c_1 e^{-2x} + c_2 e^x \).

Since \( G(x) = 4x \), we let \( y_p = Ax + B \).

\( 0 + A - 2(Ax + B) = 4x \), so \(-2Ax + (A - 2B) = 4x, A = -2 \) and \( B = -1 \). Thus \( y_p = -2x - 1 \).

The general solution to the non-homogeneous equation is \( y = y_c + y_p = c_1 e^{-2x} + c_2 e^x + -2x - 1 \).

(b) Let’s solve the complementary equation \( y''' - 2y'' + y' = 0 \), substitution of \( y = e^{rx} \) into the differential equation yields the auxiliary equation. \( r^3 - 2r^2 + r = r(r - 1)^2 = 0 \)

So \( r = 0 \) and \( r = 1 \) which is a repeated root give the complementary solution \( y_c = c_1 + c_2 e^x + c_3 xe^x \).
Since $G(x) = 1$ and the complementary solution contains constant term already. So we should choose a term containing the next higher power of $x$ as a factor. we let $y_p = Ax$.

$0 - 0 + A = 1$, so $A = 1$. Thus $y_p = x$.

The general solution to the non-homogeneous equation is $y = y_c + y_p = c_1 + c_2 e^x + c_3 x e^x + x$.

(c) $y'' - 2y' + 5y = 0$

Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation.

$r^2 - 2r + 5 = 0$

So $r = 1 + 2i$ and $r = 1 - 2i$ which are two complex roots give the general solution $y = e^x (c_1 \cos 2x + c_2 \sin 2x)$.

20 For each of these initial value problems, find the indicated value of $y$.

(a) $y' = (x^2 + y) / x$ \quad $y = 2$ when $x = 1$. Find $y$ when $x = 2$.

(b) $y'' = (yy')^3$ \quad $y = 1$ and $y' = -4$ when $x = 0$. Find $y$ when $x = 1$.

(a) $y' = \frac{x^2 + y}{x}$, $y(1) = 2$, find $y(2) = ?$

So $y' - \frac{y}{x} = x$. Thus $P(x) = -\frac{1}{x}$.

So $F(x) = e^\int -\frac{1}{x} dx = e^{-\int \frac{1}{x} dx} = e^{-\ln|x|} = (e^{\ln|x|})^{-1}$

Since $x > 0$ at the starting point, $(e^{\ln|x|})^{-1} = (e^{\ln x})^{-1} = x^{-1} = \frac{1}{x}$.

So $\frac{1}{x^2}y = \int \frac{1}{x^2} x dx = x + C$, $y = x^2 + Cx$.

Since $y(1) = 2, 2 = 1 + C$. So $C = 1$, and $y = x^2 + x$.

Thus $y(2) = 2^2 + 2 = 6$. 
(b) $y'' = (yy')^3$, $y(0) = 1, y'(0) = -4$, find $y(1) =$?

Let $P = y' = \frac{dy}{dx}$, $y'' = \frac{dP}{dx} = \frac{dP}{dy} \frac{dy}{dx} = \frac{dP}{dy} P$

$\frac{dP}{dy} P = y^3 P^3$, since $P(0) = y'(0) = -4, P \neq 0$ near starting point.

$\frac{dP}{dy} = y^3 P^2$, $\int \frac{dP}{P^2} = \int y^3 dy$.

So $- \frac{1}{P} = \frac{y^4}{4} + C$. since $P(0) = y'(0) = -4$ and $y(0) = 1, \frac{1}{4} = \frac{1}{4} + C$. So $C = 0$.

$- \frac{1}{P} = \frac{y^4}{4},$ so $\frac{dy}{dx} = P = -4 \frac{y^4}{y^4}$.

$\int y^4 dy = -4 \int dx$, $\frac{y^5}{5} = -4x + C$. So $C = \frac{1}{5}$. Hence $y^5 = -20x + 1$, $y = \sqrt[5]{-20x + 1}$.

$\text{So } y(1) = \sqrt[5]{-20 + 1} = \sqrt[5]{-19}$. 

\[\Box\]

21 The graph of the function $y = f(x)$ for $x \geq 0$ has the property that the normal line at each point $(x, y)$ on the curve crosses the $y$-axis at $(0, y/2)$. The curve crosses the $y$-axis at $(0, 4)$. Where does the curve cross the $x$-axis?

This question is asking us where an unknown curve $y = f(x)$ crosses the $x$-axis. We’re given information about the normal lines of $y$, so we should try and find a differential equation that $y$ will satisfy. Then we can solve this differential equation to find the unknown curve $y$ and compute it’s $x$ intercepts.

Let’s start by organizing our information: we’re considering an unknown function $y = f(x)$, the slope of the tangent line at each point $x$ is given by $f'(x)$, so the slope of the normal line at each point $x$ will be $-\frac{1}{f'(x)}$.

Moreover, the problem states that the normal line at each point $(x, y)$ crosses the $y$-axis at $(0, y/2)$. Now, we know the slopes of the normal lines, $-\frac{1}{f'(x)}$, and we know a point each normal line passes through, $(0, y/2)$, so we can write down the equation for the normal lines using point-slope form:

$y - \frac{y}{2} = -\frac{1}{f'(x)} (x - 0)$

$y = \frac{-x}{f'(x)}$

$f'(x)y = -2x$
This is a differential equation, which we can re-write using more familiar notation:

\[
\frac{dy}{dx} y = -2x
\]

This is a separable first order equation, which we know how to solve:

\[
\int y \frac{dy}{dx} dx = \int -2x dx \\
\int y dy = -2 \int x dx \\
\frac{1}{2} y^2 = -x^2 + c \\
y^2 = -2x^2 + c
\]

We’re given an initial condition for the function \(y\), so plugging in \(x = 4\) we get that \(c = 32\), so we’ve found the unknown curve: \(y^2 = -2x^2 + 32\). Setting \(y = 0\), we have \(32 = 2x^2\), so the curve crosses the \(x\)-axis at \(x = 4\) (we aren’t worried about \(x = -4\) since the problem states that we’re only considering values of \(x\) greater than or equal to 0).

\[
\blacksquare
\]

22. Solve these initial value problems.
(a) \(xy' - y = x^3\) \(y(2) = 6\).  (b) \(y' = (x + y)/x\) \(y(1) = 1\).

(a) Re-writing the equation in standard form:

\[
y' - \frac{y}{x} = x^2
\]

Then, our integrating factor is: \(e\int \frac{x}{x} dx = e^{-\ln x} = e^{\ln x} = \frac{1}{x}\), so we have:

\[
y' - \frac{y}{x} = x^2 \\
d \left( \frac{1}{x} y \right) = x^2 \frac{1}{x} = x \\
\int \left[ d \left( \frac{1}{x} y \right) \right] dx = \int x dx \\
y \frac{1}{x} = \frac{1}{2} x^2 + c
\]

So we have \(y \frac{1}{x} = \frac{1}{2} x^2 + c\), substituting the initial value \(y(2) = 6\) gives us \(\frac{1}{2}6 = \frac{1}{2}4 + c\), so \(c = -1\) and we have \(y = \frac{1}{2} x^3 - x\).
(b) Writing the equation in standard form, we have:

\[ y' - \frac{y}{x} = 1 \]

So our integrating factor is again \( e^{\int \frac{-1}{x} \, dx} = \frac{1}{x} \). We proceed:

\[ y' - \frac{y}{x} = 1 \]

\[ \frac{d}{dx} \left( \frac{x}{y} \right) = \frac{1}{x} \]

\[ \int \left[ \frac{d}{dx} \left( \frac{1}{x} y \right) \right] \, dx = \int \frac{1}{x} \, dx \]

\[ \frac{1}{x} y = \ln x + c \]

Plugging in the initial value, we have \( \frac{1}{1} 1 = \ln 1 + c \), so \( c = 1 \) and \( y = x \ln x + x \).

---

23 Solve this initial value problem. \( y'' = 8(y^3 + y) \). \( y(1) = 1 \), \( y'(1) = 4 \).

Consider the initial value problem \( y'' = 8(y^3 + y) \). This is a second order problem, so let’s attempt solving it by turning it into a first order problem.

Let \( p = y' \), then \( y'' \) is the second derivative of \( y \) with respect to \( x \), so \( y'' = \frac{dp}{dx} \). The chain rule gives us:

\[ \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} p \]

Now we can re-write the original differential equation as \( \frac{dp}{dy} p = 8y^3 + 8y \), which is now just a separable first order equation. Our first step is to solve for the unknown function \( p \), so note that first we want to integrate with respect to \( y \):

\[ \frac{dp}{dy} p = 8y^3 + 8y \]

\[ \int p \frac{dp}{dy} \, dy = \int 8y^3 + 8y \, dy \]

\[ \frac{1}{2} p^2 = 2y^4 + 4y^2 + c \]

\[ p^2 = 4y^4 + 8y^2 + c \]

Plugging in the initial value \( y'(1) = p(1) = 4 \) we get \( 16 = 4 + 8 + c \), so \( c = 4 \) and \( p^2 = 4y^4 + 8y^2 + 4 \).
Our next step is to “unmask” $p$ as a function of $x$ (remember $p$ is just $\frac{dy}{dx}$) and solve, but before we proceed note that right now we have an equation for $p^2$, and solving a differential equation involving $p^2 = \left(\frac{dy}{dx}\right)^2$ will be difficult, so let’s factor before proceeding:

\[
p^2 = 4y^4 + 8y^2 + 4
\]

\[
p^2 = (2y^2 + 2)^2
\]

So $p = \pm(2y^2 + 2)$. However, note that the initial condition for $y' = p$ is positive, so we can write $p = 2y^2 + 2$. Now, “unmasking” $p$, we have $p = \frac{dy}{dx} = 2y^2 + 2$. This is another separable first order equation, so we proceed. Note that for this equation, we’re integrating with respect to $x$, since we’re trying to solve an equation involving $\frac{dy}{dx}$ for the original function $y$.

\[
\frac{dy}{dx} = 2y^2 + 2
\]

\[
\frac{1}{2y^2 + 2} \frac{dy}{dx} = 1
\]

\[
\int \frac{1}{2y^2 + 2} \frac{dy}{dx} dx = \int 1 dx
\]

\[
\frac{1}{2} \int \frac{dx}{y^2 + 1} = x + c
\]

\[
\frac{1}{2} \tan^{-1} y = x + c
\]

\[
\tan^{-1} y = 2x + c
\]

The remaining initial condition allows us to write $\tan^{-1} 1 = 2 \cdot 1 + c$, so we have $\frac{\pi}{4} = 2 + c$, so $c = \frac{\pi - 8}{4}$ and we have $y = \tan(2x + \frac{\pi - 8}{4})$.

**Verify:** Since this problem was a bit more difficult, it’s not a bad idea to check our answer. First, note that $y(1) = \tan(2 \cdot 1 + \frac{\pi - 8}{4}) = \tan(2 + \frac{\pi}{4} - 2) = \tan(\frac{\pi}{4}) = 1$, so our answer certainly satisfies the initial condition.

Now, if $y = \tan(2x + \frac{\pi - 8}{4})$, then $y' = 2 \sec^2(2x + \frac{\pi - 8}{4})$ and $y'' = 8 \sec^2(2x + \frac{\pi - 8}{4}) \tan(2x + \frac{\pi - 8}{4})$. We want to verify that this $y$, $y'$, and $y''$ satisfy the original differential equation $y'' = 8(y^3 + y)$:

\[
8(y^3 + y) = 8(\tan^3(2x + \frac{\pi - 8}{4}) + \tan(2x + \frac{\pi - 8}{4}))
\]

\[
= 8(\tan(2x + \frac{\pi - 8}{4})[\tan^2(2x + \frac{\pi - 8}{4}) + 1])
\]

\[
= 8(\tan(2x + \frac{\pi - 8}{4})[\sec^2(2x + \frac{\pi - 8}{4})])
\]

According to the differential equation the last line should equal $y''$, and in fact it does! ■
Let $L$ be the linear differential operator given by the formula $L(y) = x^2y'' - 2xy' + 2y$.

(a) Compute $L(x)$, $L(x^2)$ and $L(x^3)$.  

(b) Find the general solution for $x^2y'' - 2xy' + 2y = 4x^3$.

(a) This is just plug ‘n chug - as long as you keep your cool:

$L(x) = x^2(x)' - 2x(x)' + 2(x)$
\[ = 0 - 2x + 2x \]
\[ = 0 \]

$L(x^2) = x^2(x^2)' - 2x(x^2)' + 2(x^2)$
\[ = x^2(2) - 4x^2 + 2x^2 \]
\[ = 0 \]

$L(x^3) = x^2(x^3)' - 2x(x^3)' + 2(x^3)$
\[ = 6x^3 - 6x^3 + 2x^3 \]
\[ = 2x^3 \]

(b) Consider the differential equation $x^2y'' - 2xy' + 2y = 4x^3$. Note that a solution $y$ to this differential equation will also satisfy $L(y) = 4x^3$.

Now, recall that in part (a) we found that $L(x^3) = 2x^3$, so if $y = 2x^3$, then $L(2x^3) = 2L(x^3) = 2(2x^3) = 4x^3$, so $y = 2x^3$ is a solution since $L(x) = L(x^2) = 0$, a solution $y$ can have any number of $x$ and $x^2$ terms added without changing $L(y)$. For example,

$L(2x^3 + 4x^2 + 15x) = L(2x^3) + L(4x^2) + L(15x)$
\[ = 2L(x^3) + 4L(x^2) + 15L(x) \]
\[ = 4x^3 + 0 + 0 \]
\[ = 4x^3 \]

This implies that the general solution is given by $y = 2x^3 + c_1x^2 + c_2x$, where $c_1$ and $c_2$ are any real numbers.
Integrate. (a) \[ \int_{0}^{1} \frac{x + 1}{x^2 + 2x - 4} \, dx \] (b) \[ \int x \sec^2(x) \, dx \] (c) \[ \int \cos^4 \left( \frac{x}{4} \right) \, dx \]

(a) It appears that the denominator here is inviting us to complete the square, and that will most likely work, but let’s be a little sneaky:

\[ I = \int_{0}^{1} \frac{x + 1}{x^2 + 2x - 4} \, dx \quad u = x^2 + 2x - 4 \]

\[ du = 2x + 2 = 2(x + 1) \]

\[ = \frac{1}{2} \int_{-4}^{-1} \frac{du}{u} \]

\[ = \frac{1}{2} \left[ \ln |u| \right]_{-4}^{-1} \]

\[ = \frac{1}{2} (\ln 1 - \ln 4) \]

\[ = \frac{1}{2} (-\ln 4) \]

\[ = -\ln 2 \]

(b) This looks like some integration by parts:

\[ I = \int x \sec^2 x \, dx \]

\[ u = x \quad du = dx \]

\[ dv = \sec^2 x \, dx \quad v = \tan x \]

\[ I = x \tan x - \int \tan x \, dx \]

\[ = x \tan x + \ln | \cos x | + c \]
(c) We don’t know many identities involving powers of cosine, so let’s see where we can get:

\[
\int \cos^4 \left( \frac{x}{4} \right) \, dx
\]
\[
= \int \left( \cos^2 \left( \frac{x}{4} \right) \right)^2 \, dx
\]
\[
= \int \left( \frac{1}{2} \left( 1 + \cos \left( \frac{x}{2} \right) \right) \right)^2 \, dx
\]
\[
= \frac{1}{4} \int \left( 1 + \cos \left( \frac{x}{2} \right) \right)^2 \, dx
\]
\[
= \frac{1}{4} \int \left[ 1 + 2 \cos \left( \frac{x}{2} \right) + \cos^2 \left( \frac{x}{2} \right) \right] \, dx
\]
\[
= \frac{1}{4} x + \sin \left( \frac{x}{2} \right) + \frac{1}{4} \int \cos^2 \left( \frac{x}{2} \right) \, dx
\]
\[
= \frac{1}{4} x + \sin \left( \frac{x}{2} \right) + \frac{1}{4} \cdot \frac{1}{2} (1 + \cos x) \, dx
\]
\[
= \frac{3x}{8} + \sin \left( \frac{x}{2} \right) + \frac{\sin x}{8} + c
\]

26 Integrate \( \int_2^4 \frac{\sqrt{x^2 - 4}}{x} \, dx \).

A square root is in our way, so let’s try a trig substitution.

\[
I = \int_2^4 \frac{\sqrt{x^2 - 4}}{x} \, dx
\]
\[
x = 2 \sec \theta \quad \quad \quad x = 2 \rightarrow \theta = \sec^{-1}(1) = 0
\]
\[
dx = 2 \sec \theta \tan \theta \, d\theta \quad \quad \quad x = 4 \rightarrow \theta = \sec^{-1}(2) = \frac{\pi}{3}
\]
\[
I = \int_0^{\pi/3} \frac{\sqrt{4 \sec^2 \theta - 4(2 \sec \theta \tan \theta)} \, d\theta}{2 \sec \theta}
\]
\[
= 2 \int_0^{\pi/3} \tan^2 \theta \, d\theta
\]
\[
= 2 \int_0^{\pi/3} \sec^2 \theta - 1 \, d\theta
\]
\[
= 2 [\tan \theta - \theta]_0^{\pi/3}
\]
\[
= 2 \left( \sqrt{3} - \frac{\pi}{3} \right)
\]
\[
= 2\sqrt{3} - \frac{2\pi}{3}
\]
Use Simpson’s rule with \( n = 4 \) to obtain a numerical approximation for \( \int_1^3 \frac{dx}{1+x} \).

The interval of integration has length \( 3 - 1 = 2 \), so with \( n = 4 \) steps our \( \Delta x \) for each step will be \( \frac{1}{2} \). The function we’re approximating is \( f(x) = \frac{1}{1+x} \). Simpson’s rule states that the integral above will be approximated by:

\[
I \approx \frac{1/2}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)
\]

Where the \( y_i \)'s are the values of \( f(x) \) evaluated at the various intervals. Substituting this all in, we have:

\[
I \approx \frac{1/2}{3} (f(1) + 4f(1 + \frac{1}{2}) + 2f(1 + \frac{3}{2}) + 4f(1 + \frac{5}{2}) + f(1 + \frac{7}{2}))
\]

\[
I \approx \frac{1/2}{3} (f(1) + 4f(3/2) + 2f(2) + 4f(\frac{5}{2}) + f(3))
\]

\[
I \approx \frac{1/2}{3} \left( \frac{1}{2} + \frac{8}{5} + \frac{2}{3} + \frac{8}{7} + \frac{1}{4} \right)
\]

\[
\approx \frac{1}{6} \left( \frac{1747}{420} \right) \approx 0.69325
\]

Mathematica gives a numerical approximation to the answer as 0.69314718..., so our approximation is pretty good!
Integrate. (a) \( \int_{1}^{\sqrt{2}} \frac{dx}{\sqrt{2x^2 - 1}} \). (b) \( \int x^2 \ln(x) \, dx \).

(a) As usual, let’s use a trig substitution to get rid of an unwanted square root:

\[
I = \int_{1}^{\sqrt{2}} \frac{dx}{\sqrt{2x^2 - 1}}
\]

\(2x^2 = \sec^2 \theta\)

\[x = \frac{1}{\sqrt{2}} \sec \theta\]

\[dx = \frac{1}{\sqrt{2}} \sec \theta \tan \theta \, d\theta\]

\[
I = \int_{\pi/4}^{\pi/3} \frac{1}{\sqrt{2}} \sec \theta \tan \theta \, d\theta
\]

\[
= \frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/3} \frac{\sec \theta \tan \theta \, d\theta}{\sqrt{\sec^2 \theta - 1}}
\]

\[
= \frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/3} \frac{\sec \theta \, d\theta}{\tan \theta}
\]

\[
= \frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/3} \sec \theta \, d\theta
\]

\[
= \frac{1}{\sqrt{2}} \left[ \ln(\sec \theta + \tan \theta) \right]_{\pi/4}^{\pi/3}
\]

\[
= \frac{1}{\sqrt{2}} \left( \ln(2 + \sqrt{3}) - \ln(\sqrt{2} + 1) \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( \ln \frac{2 + \sqrt{3}}{1 + \sqrt{2}} \right)
\]

(b) Integration by parts. Recall that when integrating by parts involving \( \ln \), generally we try setting “\( u \)” to \( \ln \) (since integrating \( \ln \) is a bit messy)

\[
I = \int x^2 \ln(x) \, dx
\]

\[u = \ln(x)\]

\[du = \frac{1}{x} \, dx\]

\[dv = x^2 \, dx\]

\[v = \frac{1}{3} x^3\]

\[
I = \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \frac{1}{x} \, dx
\]

\[
= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 \, dx
\]

\[
= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c
\]
29 Integrate. (a) \( \int \frac{4x^4 + 1}{4x^3 - x} \, dx \). (b) \( \int \sin(\ln(x)) \, dx \).

(a) Performing long division on the integrand, we have \( \frac{4x^4 + 1}{4x^3 - x} = x + \frac{x^2 + 1}{4x^3 - x} \), so we start with \( \int x + \frac{x^2 + 1}{4x^3 - x} \, dx \). Writing the fraction in the integrand as \( \frac{x^2 + 1}{x(2x + 1)(2x - 1)} \), we can write down a partial fraction expansion:

\[
\frac{x^2 + 1}{x(2x + 1)(2x - 1)} = \frac{A}{x} + \frac{B}{2x + 1} + \frac{C}{2x - 1}
\]

Using the cover-up method, we can quickly find \( A = -1 \), \( B = C = \frac{5}{4} \), so our integral becomes:

\[
I = \int x + \frac{x^2 + 1}{4x^3 - x} \, dx = \int x \, dx + \int \frac{-dx}{x} + \frac{5}{4} \int \frac{dx}{2x + 1} + \frac{5}{4} \int \frac{dx}{2x - 1}
\]

\[
= \frac{x^2}{2} - \ln |x| + \frac{5}{8} \left( \ln |2x + 1| + \ln |2x - 1| \right) + c
\]

(b) Let’s use a \( u \)-substitution to convert this problem into a friendlier form:

\[
I = \int \sin(\ln(x)) \, dx
\]

\[
u = \ln(x) \quad x = e^u
\]

\[
du = \frac{1}{x} \, dx \quad e^u \, du = dx
\]

\[
I = \int \sin(u) e^u \, du
\]

Now we proceed using integration by parts:

\[
I = \int \sin(u) e^u \, du
\]

\[
r = e^u \quad dr = e^u \, du
\]

\[
ds = \sin(u) \, du \quad s = -\cos(u)
\]

\[
I = -\cos(u) e^u + \int \cos(u) e^u \, du
\]
Despite the fact that the remaining integral is tricky, we keep our cool and try again:

\[
I = -\cos(u)e^u + \int \cos(u)e^u du \\
r = e^u \\
ds = \cos(u) du \\
\int \sin(u)e^u du = -\cos(u)e^u + e^u \sin(u) - \int \sin(u)e^u du \\
2 \int \sin(u)e^u du = e^u (\sin(u) - \cos(u)) \\
\int \sin(u)e^u du = \frac{e^u (\sin(u) - \cos(u))}{2}
\]

30 Do well on the exam!

You can do it!