Math 660-HW3

1. Determine which one of the following two Fourier spectral methods for computing derivatives is better:
   - Given $v = (v_1, \ldots, v_N)$, compute $\hat{v}$.
   - Define $\hat{w}_k = (ik)^m \hat{v}_k$, for $k = -N/2+1, \ldots, N/2$ (or $k = 0, \ldots, N/2, -N/2+1, \ldots, -1$ in Matlab).
   - Compute the inverse DFT (inverse FFT) and get $w_k$, the real part of which is the approximation of the derivative.

   and

   - Given $v = (v_1, \ldots, v_N)$, compute $\hat{v}$.
   - Define $\hat{w}_k = (ik)^m \hat{v}_k$, for $k = 0, 1, \ldots, N$.
   - Compute the inverse DFT (inverse FFT) and get $w_k$, the real part of which is the approximation of the derivative.

   Explain why that is true.

2. Consider the Stokes equation

   $-\Delta u + \nabla p = f$
   $\nabla \cdot u = 0$

   on $[0, 1] \times [0, 1]$ with periodic boundary conditions. Here, $f$ is the external force satisfying $\iint f \, dx \, dy = 0$. $u$ is the velocity field and $p$ is the pressure. Find the velocity with zero mean using Fourier spectral method. Plot the velocity field using ‘quiver’ command for $f = \langle \sin(2\pi x)^2 - \frac{1}{2}, 0 \rangle$ and $\langle \sin(2\pi y), \cos(2\pi x) \rangle$.

3. Consider the Schrodinger equation

   $i\epsilon u_t + \frac{1}{2} \epsilon^2 u_{xx} - V(x)u = 0$
   $u(x, 0) = u_0(x)$

   Using the time-splitting method, you can solve the following two equations

   $i\epsilon u_t + \frac{1}{2} \epsilon^2 u_{xx} = 0$
   $i\epsilon u_t - V(x)u = 0$
from $t^n$ to $t^{n+1}$. The first equation can be solved using Fourier spectral method while the second one can be solved by hand.

The density of probability is given by

$$n = |u|^2.$$ 

For

$$u_0 = e^{-5(x-1/2)^2 + i \ln(e^{x-1} + e^{-(x-1)})/\epsilon},$$

and $V(x) = 5$, compute the density at $t = 0.3$ for $\epsilon = 0.0008$ with various spacial steps.