Math 660-Lecture 23

1 An example of DG

If we develop Galerkin’s method using globally continuous function \( v \), then we will get a full matrix. The numerical solution can be solved all at once. For the elliptical equations, the solution for every point is obtained all at once. This makes sense.

However, for stationary hyperbolic problem, there is a natural characteristics, i.e. the direction of propagation of information. In PDE, the causal relation allows us to solve the value one by one. If we use Galerkin’s method, we will have to solve the big matrix. This is not convenient. We hope to decouple the weak formulation and solve the linear system following the propagation of information. This then motivates the so-called discontinuous Galerkin method (DG) may be applied.

Consider the example, which is hyperbolic:

\[
\beta \cdot \nabla u + u = f \\
u|_{\Gamma_-} = g \quad \Gamma_- = \{ x : \hat{n} \cdot \beta < 0 \}
\]

We now develop a DG for this problem in which the weak solution is only in \( L^2 \).

1.1 FEM

Consider a given triangulation \( T_h = \{ K \} \). Here let’s agree that \( K \) only contains the interior. For each triangle \( K \), define \( \partial K_- = \{ \hat{n} \cdot \beta < 0 \} \) and \( \partial K_+ \). Define \( U_+ = \lim_{s \to 0^+} U(x + s\beta) \) and similarly \( U_- \). If \( \partial K_- \subset \Gamma_- \), we define \( u^-(x) = g(x) \). (Draw a picture to see.)

The weak formulation is given by

\[
B(u, v) := \sum_K B_K(u, v) = (f, v) = \sum_K (f, v), \quad \forall v \in L^2(\Omega), \text{continuous on } K,
\]

where

\[
B_K(u, v) := (\beta \cdot \nabla u + u, v)_K - \int_{\partial K_-} [u]v_+ \hat{n} \cdot \beta ds.
\]

Note that both \( u \) and \( v \) can be discontinuous from one triangle to another one. To understand this weak formulation, one way is to think about \( D_K = K \cup \partial K_- \) and we define the value of \( u \) on \( \partial K_- \) to be \( u^- \) (the value
on $\Gamma_+$ is un-defined). Then, we multiply $v$ that is continuous on $D_K$ (but may be discontinuous from one $D_K$ to another) and integrate. Then, the integral on $\partial K_-$ is mostly zero except for $\beta \cdot \nabla uv$ term. $u$ has jump, and $\nabla u$ is a distribution here. This delta-like distribution is integrated to the last term on the left hand side.

The above is formal understanding. Now, we do this rigorously, which is essentially the above understanding. We assume $u$ is smooth on $K$ but may be discontinuous on the boundary of each triangle. According to the PDE: $v \in C^1(\Omega)$:

$$-\int_{\Omega} u\beta \cdot \nabla v \, dx + \int_{\partial \Omega} \beta \cdot \hat{n}uv \, ds + \int_{\Omega} uv \, dx = \int_{\Omega} fv \, dx.$$ 

Since $u \in L^2(\Omega)$, we can break the integrals into integrals on each $K$. For example, the first becomes

$$-\sum_K \int_K u\beta \cdot \nabla v \, dx = \sum_K \int_K (\beta \nabla \cdot u)v \, dx - \int_{\partial K} \beta \cdot \hat{n}u_{in}v_{in} \, ds$$

Here, $u_{in}$ means the inner limit. Clearly, on $\partial K_+\!, \, u_{in} = u_+$ while on $\partial K_-\!, \, u_{in} = u_-$. For convenience, we define $u_+ = u_-$ on $\Gamma_+$. Then, we have

$$-\int_{\Omega} u\beta \cdot \nabla v \, dx + \int_{\partial \Omega} \beta \cdot \hat{n}uv \, ds = \sum_K (\beta \cdot \nabla u + u, v)_K - \int_{\partial K_-} [u]v_+\hat{n} \cdot \beta \, ds.$$ 

Now, we can talk about the weak formulation on each $K$. However, as soon as we focus on each $K$, we do not need the information of $v$ across the boundary, so we can only assume that $v$ is smooth on $K$ and use $v_+$ on boundary $\partial K_-$. 

For the finite element space, the function is still a piecewise polynomial on each triangle but no continuity requirement is enforced at the interface.

$$V_h = \{ v \in L^2 | v_K \in P_r(K) \}$$

We then find solutions in $V_h$. In our case, we can simply let $r = 0$, namely the function is a constant on each triangle and replace $u, v$ with $u_h, v_h \in V_h$. We then have the DG formulation. The solution process begins with those triangles near $\Gamma_-$. After the solution on these triangles are obtained, the triangles next to them can be solved. This process can then be moved on.

Let’s look at 1D case of this problem: consider 1D and $\beta = 1$. $u_x + u = f$. Suppose $u_h$ is constant $u_j$ on the interval $I_j = (x_j, x_{j+1})$. Then, the
formulation on $I_j$ is reduced to

$$(0 + u_j v_j h_j) + (u_j - u_{j-1}) v_j = v_j \int_{I_j} f \, dx$$

which implies $\frac{u_j - u_{j-1}}{h_j} + u_j = \frac{1}{h_j} \int_{I_j} f \, dx$. This is the upwind scheme.

1.2 Error estimate of the DG

We will assume that the solution of the PDE $u$ is nice. Then, clearly,

$$B(u, v) = (f, v), \forall v \in V_h.$$

Then, introducing $e = u - u_h$, we have

$$B(e, v) = 0, \forall v \in V_h.$$

Let’s now compute $B(v, v)$. Integrating by parts,

$$(\beta \cdot \nabla v, v)_K = \int_{\partial K} v^2 \hat{n} \cdot \beta \, ds - (\beta \cdot \nabla v, v)_K$$

yields

$$2(\beta \cdot \nabla v, v)_K = \int_{\partial K_+} v^2 \hat{n} \cdot \beta \, ds - \int_{\partial K_-} v^2 |\hat{n} \cdot \beta| \, ds.$$

Hence, we find that

$$2B(v, v) = 2(v, v) + \sum \int_{\partial K_+} v^2 \hat{n} \cdot \beta \, ds - \sum \int_{\partial K_-} v^2 |\hat{n} \cdot \beta| \, ds$$

$$- 2 \int_{\partial K_-} (v_+ - v_-) v_+ \hat{n} \cdot \beta \, ds = 2(v, v) + \sum \int_{\partial K_-} v^2 |\hat{n} \cdot \beta| \, ds$$

$$+ \int_{\Gamma_+} v^2 |\hat{n} \cdot \beta| \, ds - \int_{\Gamma_-} v^2 |\hat{n} \cdot \beta| \, ds$$

$$- \sum \int_{\partial K_-} v^2 |\hat{n} \cdot \beta| \, ds - 2 \int_{\partial K_-} (v_+ - v_-) v_+ \hat{n} \cdot \beta \, ds$$

$$= 2||v||^2 + \sum_K \int_{\partial K_-} [v]^2 |\hat{n} \cdot \beta| \, ds + \int_{\Gamma_+} v^2 |\hat{n} \cdot \beta| \, ds - \int_{\Gamma_-} v^2 |\hat{n} \cdot \beta| \, ds$$

$$= 2||v||^2 - \int_{\Gamma_-} v^2 |\hat{n} \cdot \beta| \, ds.$$
Now, we estimate the error. For this purpose, we define
\[ \pi_h u|_K = \frac{1}{|K|} \int_K u \, dx. \]

Then, there is the error estimate

**Theorem 1.**
\[ \| e \|_\beta \leq Ch^{1/2} \| u \|_{H^1} \]

for \( r = 0 \).

**Proof.** Let \( \eta^h = u - \pi_h u \). Since \( e_{\Gamma} = 0 \), we have
\[ \| e \|_\beta^2 = B(e, e) = B(e, u - \pi_h u) = B(e, \eta^h) = \sum_K (\beta \cdot \nabla e + e, \eta^h)_K - \int_{\partial K_0} [e] \eta^h n \cdot \beta ds \]

On each triangle, \( \nabla e = \nabla u \) because \( \nabla u_h = 0 \). Then, we have
\[ \frac{1}{2} \sum_K \sqrt{\int_{\partial K_0} |e|^2 |n \cdot \beta| ds} + \frac{1}{2} \sum_K \sqrt{\int_{\partial K_0} (\eta^h)^2 |n \cdot \beta| ds} \]

The second term is \( \leq \frac{1}{2} \| e \|_{L^2}^2 + \frac{1}{2} \| \eta^h \|_{L^2}^2 \). The third term
\[ \leq \frac{1}{4} \sum_K \int_{\partial K_0} [e]^2 |n \cdot \beta| ds + \sum_K \int_{\partial K_0} (\eta^h)^2 |n \cdot \beta| ds \]

Hence,
\[ \frac{1}{2} \| e \|_\beta^2 \leq \| \beta \cdot \nabla u \|_{L^2} \| \eta^h \|_{L^2} + \frac{1}{2} \| \eta^h \|_{L^2}^2 + \sum_K \int_{\partial K_0} (\eta^h)^2 |n \cdot \beta| ds \]

Finally, using the fact
\[ \| \eta^h \|_K \leq Ch |u|_{H^{1/2}(K)} \]
we have \( \| \eta^h \|_{L^2} \leq C \| u \|_{H^1} h \). The claim the follows. \( \square \)

2 **Boundary element method for Laplace equation**

(Chapter 10 in Johnson’s book.)
2.1 The theory and ideas

Consider the following exterior problem:

\[-\Delta u = 0 \quad x \in \Omega' = \mathbb{R}^3 \setminus \bar{\Omega} \]

\[u_{\Gamma} = u_0 \quad u \to 0, \quad |x| \to \infty\]

This equation has applications in aerodynamics, underwater acoustics. The first way is to truncate the domain using a large ball and solve directly. Another way is to reformulate it into an integral equation on \(\Gamma = \partial \Omega\) and then apply the boundary integral method.

The fundamental solution of the Laplace equation is

\[E(x) = \frac{1}{4\pi|x|}\]

which satisfies \(-\Delta E(x) = \delta(x)\). Note that again we have a different sign from the book because we used \(-\Delta\) instead of \(\Delta\).

**Theorem 2.** \(u\) is smooth on \(\mathbb{R}^3 \setminus \Gamma\) and \(\Delta u = 0\) for \(x \notin \Gamma\). Then, as \(|x| \to \infty\), \(u = O(|x|^{-1}), |\nabla u| = O(|x|^{-2})\). Further, letting \(u^i\) be the one in \(\Omega\) and \(u^e(x)\) be the one in \(\Omega'\) (the interior minus the exterior), denoting \([u] = u^i - u^e\) for \(x \in \Gamma\), we have

\[\frac{1}{4\pi} \int_{\Gamma} \left( \frac{1}{|x - y|} \frac{\partial u}{\partial n} - \frac{1}{|x - y|} \frac{\partial}{\partial n} (\frac{1}{|x - y|}) \right) dS(y) = \begin{cases} u(x) & x \notin \Gamma \\ (u^i + u^e)/2 & x \in \Gamma \end{cases}\]

where \(n\) is the outer normal of \(\Omega\) (pointing into the exterior domain).

Note that \(-\Delta u = 0\) doesn’t hold on \(\Gamma\). If it holds on \(\Gamma\), then the solution will be trivially 0.

**Proof.** Let’s consider \(x \in \Omega\) first. Then, we let \(\Omega_\epsilon = \Omega \setminus B(x, \epsilon)\). Applying Green’s Theorem on this domain

\[0 = \int_{\Omega_\epsilon} (v \Delta u^i - u^i \Delta v) dy = \int_{\partial \Omega_\epsilon} \left( v \frac{\partial u^i}{\partial n} - u^i \frac{\partial v}{\partial n} \right) dS(y)\]

where \(v = E(y - x)\). For the inner boundary \(\partial B(x, \epsilon)\), as \(\epsilon \to 0\), only the integral of \(-u^i \partial v/\partial n\) is nonzero which gives \(-u^i(x)\). Hence, we have

\[0 = \int_{\Gamma} (E(y - x) \frac{\partial u^i}{\partial n} - u^i \frac{\partial E(y - x)}{\partial n}) dS(y) - u^i(x)\]
Similarly, we can apply Green’s theorem to the external domain and, noticing \( x \in \Omega \), have

\[
0 = \int_{\Gamma} (E(y - x) \frac{\partial u^e}{\partial (-n)} - u^e \frac{\partial E(y - x)}{\partial (-n)}) dS(y)
\]

Adding them together, we have the equality for \( x \in \Omega \). For \( x \in \Omega' \), it’s similar.

The only tricky part is when \( x \in \Gamma \). We actually have

\[
0 = \int_{\Gamma} (E(y - x) \frac{\partial u^i}{\partial n} - u^i \frac{\partial E(y - x)}{\partial n}) dS(y) - \frac{1}{2} u^i(x)
\]

\[
0 = \int_{\Gamma} (E(y - x) \frac{\partial u^e}{\partial (-n)} - u^e \frac{\partial E(y - x)}{\partial (-n)}) dS(y) - \frac{1}{2} u^e(x)
\]

\[\square\]

In the above \([\partial u/\partial n]\) is the density of single layer, or the density of source, which makes the force or \( \nabla u \) discontinuous but \( u \) itself is continuous. \([u]\) is the density of double layer, or the density of source dipole.

\(-[u]\) is then understood as the double layer density, while \( \frac{\partial}{\partial n_y} (\frac{1}{|x-y|}) \) is the double layer potential (kind of like the concept of dipole).

It’s possible to make the integral using only one type of singularity density. The boundary integral problem can then be solved in two ways.