Math 660-Lecture 20: Finite element spaces (II)

(Chapter 3, 4.2, 4.3)

1 2D spaces: Triangular elements

$\Omega \subset \mathbb{R}^2$ and $T_h = \{K\}$ is a collection of triangles for the triangulation.

$P_r(K) = \{v| v \text{ is a polynomial of deg} \leq r \text{ on } K\}$.

Lemma 1. $\dim(P_r) = \binom{r+2}{2} = \frac{(r+2)(r+1)}{2}$

Proof. The basis is $x_1^i x_2^j, 0 \leq i + j \leq r$. Consider we divide $r$ units into 3 distinct groups. Let’s say the number of the units in the first group is $i$ and number of units in the second group is $j$. Clearly, one such division corresponds to uniquely an $(i,j)$ pair. The number of such divisions are equivalent to choosing two from $r+2$ positions and the chosen two positions will be the partition boundary.

1.1 Examples of finite element spaces

- Consider the following finite element space:

$$V_h = \{v : \Omega \to \mathbb{R} : v|_K \in P_1(K) \text{ and well-defined at vertices}\}$$

Since $\dim(P_1(K)) = 3$, then there are 3 degrees of freedom. Let $a^i, 1 \leq i \leq 3$ be the vertices of $K$. Then, we claim:

**Theorem 1.** Given $\alpha_i, 1 \leq i \leq 3$, there is uniquely a function $v \in P_1(K)$ such that $v(a^i) = \alpha_i$. In other words, the function on $K$ is uniquely determined by its values at the vertices.

Proof. The number of unknowns (dimension, which is 3) equals the number of conditions given. Hence, we have a 3 by 3 linear system. For such a system, the solution exists uniquely is equivalent to saying the homogeneous solution is trivial. In other words, it suffices to show that if $v = C_1 + C_2 x_1 + C_3 x_2$ is zero on $a^i$, then $v = 0$.

The line $a^1 a^2$ has an equation of the form $d_1 x_1 + d_2 x_2 + d_3 = 0$. Since $v$ vanishes on this edge, we must have $v = C(x)(d_1 x_1 + d_2 x_2 + d_3)$. The degree of $v$ is at most 1, and therefore $C(x) = C$. Further, $a^3$ is not on the line, we must have $d_1 a_1^3 + d_2 a_2^3 + d_3 \neq 0$. Hence, $C = 0$. 

\[ \Box \]
Let’s consider constructing the basis functions of $V_h$. By the theorem just proved, we need to specify the values at the nodes only. Let $\lambda_i \in P_1(K)$ and $\lambda_i(a^j) = \delta_{ij}$. $\lambda_1 = \mu(d_1x_1 + d_2x_2 + d_3)$ where $d_1x_1 + \ldots$ is the equation of $a^2a^3$. Then, $\mu$ is determined uniquely by the condition $\lambda_1(a^1) = 1$.

Now, suppose we are given $v(a^i)$. We define the finite element space to be

$$V_h = \{ v : v|_K = \sum_{i=1}^{3} v(a^i)\lambda_i \}.$$  

**Theorem 2.** $V_h \subset C^0(\Omega)$.

By this definition, $w \in V_h$ is continuous at the nodes. Consider one edge $e$. Suppose $e$ is the intersection between $K_1$ and $K_2$. On $e$, we define $g = w|_{K_1} - w|_{K_2}$. $g = c_1x_1 + c_2x_2 + c_3$ is linear on $e$. It is zero at the endpoints. Then, it must be zero on the whole line containing $e$ since it is linear. ($g$ itself may not be a zero function.) Hence, $w$ is continuous at the edge.

- The dimension of $P_2(K)$ is 6. Consider

$$V_h = \{ v : v|_K \in P_2(K), \forall K \in T_h, \text{ value at } a^i \text{ and } a^{ij} \text{ are specified.} \}$$

where $a^{ij} = \frac{1}{2}(a^i + a^j)$ be the midpoint of $a^i$ and $a^j$.

We first of all have the following observation:

**Theorem 3.** $v \in P_2(K)$ is uniquely determined by the values at $a^i, 1 \leq i \leq 3$ and $a^{ij}, i < j$.

**Proof.** Again, the unknowns (dimension) equals the conditions. It suffices to show that if $v$ vanishes at these points, then it’s zero.

First of all, it vanishes on $a^1a^2$ because a 1D quadratic function is zero if it’s zero on three points. Then, $v|_{a^1a^2} = h(x)\lambda_3$ where $\lambda_3$ is the basis function in the previous example (linear function and is only 1 at $a^3$). Since $v$ is quadratic, $h$ is linear. Since $v$ also vanishes on $a^1a^3$, then $h$ vanishes on $a^1a^3$. Hence, $h = C\lambda_2$. $v = C\lambda_2\lambda_3$. Finally, $v$ also vanishes on $a^{23}$, then $C$ has to be zero.
Clearly, $\lambda_i(2\lambda_i - 1)$ is a quadratic function that is only 1 at $a^i$ and vanishes at the other five points. $4\lambda_i\lambda_j$ is the quadratic function that is 1 only at $a^{ij}$. Hence, in general,

$$v|_K = \sum_{i=1}^{3} v(a^i)\lambda_i(2\lambda_i - 1) + \sum_{i<j} v(a^{ij})4\lambda_i\lambda_j$$

Then, we actually only have $V_h \subset C^0(\Omega)$. $V_h$ is not in $C^1$. $P_2(K)$ is not enough for us to make $C^1$ functions.

- To have $C^1$ functions, we actually need $P_5(K)$. $\dim(P_5(K)) = 21$.

**Theorem 4.** $v \in P_5(K)$ is uniquely determined by $D^\alpha v(a^i), |\alpha| \leq 2$ and $\partial v(a^{ij})/\partial n$.

**Proof.** Let’s show that if on $K$, for $v$ these values are zero, then $v = 0$.

On $a^2a^3$, noting first that $v = \partial v/\partial s = \partial^2 v/\partial s^2 = 0$ at $a^2$ and $a^3$ (where $s$ is the arc length of $a^2a^3$), we conclude that $v = 0$ on this edge since there are 6 conditions for a five degree 1D polynomial.

Secondly, $g = \partial v/\partial n$ as a function on $a^2a^3$ is a polynomial of degree at most 4. $g(a^{23}) = 0$ by the given condition. $g = \partial g/\partial s = 0$ at $a^2$ and $a^3$ by the fact that $D^\alpha v = 0$. Hence $g = \partial v/\partial n$ should be zero. (we have used 10 conditions).

Then, we must have that $v = \lambda_1^2 h$. Similarly, $\lambda_2^2, \lambda_3^2$ can divide $v$ as well. These polynomials don’t have common factors. Then, $\lambda_1^2\lambda_2^2\lambda_3^2$ can divide $v$. Since $v$ is of only 5 degree, we must have $v = 0$.

Using what we have proved, we can construct $V_h = \{ v : v|_K \in P_5(K) \}$ such that if $w \in V_h$, then $w$ has continuous $D^\alpha w$ at the vertices and continuous $\partial w/\partial n$ at the midpoints. Then, we can show that $w$ is $C^1$ on one edge. Then, $V_h \subset C^1(\Omega)$.

## 2 Interpolation errors

In this lecture, we study the interpolation error of the finite element spaces studied in the previous lecture. This then guarantees the consistency of FEM for solving PDEs.
3 2D spaces: Triangular elements

For some general results, one can refer to Bramble-Hilbert lemma. Anyhow, we first of all note Note that the
\[\|v\|_{H^s(\Omega)}^2 = \sum_{K \in T_h} \|v\|_{H^s(K)}^2\]
is true provided that \(v \in H^s(\Omega)\). Hence, it reduces to considering one element.

**Remark 1.** For \(v \in H^s(\Omega)\) made from \(P_r(K)\) (piecewise polynomial on each element), we need \(D^\alpha v (|\alpha| < s)\) to be continuous on \(\Omega\).

We prove some lemmas for \(P_1(K)\) functions first. We first define two quantities:
\[h_K = \text{The diameter of } K\]
\[\rho_K = \text{The diameter of the inscribed circle}\]
Let \(\beta = \inf_K \rho_K / h_K\). We assume that \(\beta\) is bounded below, which requires that all triangles are kind of regular.

### 3.1 The approximations errors on one element

First of all, we have the following:

**Lemma 2.** Let \(\lambda_i\) be the linear functions as above \((\lambda_i(a^j) = \delta_{ij})\). Then, they form a basis of \(P_1(K)\). Further,
\[
\sum_{i=1}^{3} \lambda_i = 1 \\
\sum_{i=1}^{3} a^i_j \lambda_i(x) = x_j, j = 1, 2
\]
It’s clear that they should be true because \(\lambda_i\)'s make up a basis for linear functions. Then, any linear function can be written as the linear combination of them.

**Remark 2.** This lemma says that the interpolation for constant and linear functions should be exact.
For a smooth function $v(x)$, let’s define

$$p_i(x) = \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x)(a^i_j - x_j),$$

where $a^i = (a^i_1, a^i_2)$. Then, we have

**Lemma 3.**

$$\sum_{i=1}^{3} p_i(x) \lambda_i(x) = 0. \forall x \in K.$$  
$$\sum_{i=1}^{3} p_i(x) \frac{\partial \lambda_i}{\partial x_j} = \frac{\partial v}{\partial x_j}(x)$$

The first claim follows from $\sum_{i}(a^i_j - x_j)\lambda_i = 0$.

For the second claim, we note

$$\sum_{i} \sum_{k} \frac{\partial v}{\partial x_k}(-x_k)\frac{\partial \lambda_i}{\partial x_j} = \sum_{k} \frac{\partial v}{\partial x_k}(-x_k)\frac{\partial \sum_{i} \lambda_i}{\partial x_j} = 0.$$  
$$\sum_{i} \sum_{k} \frac{\partial v}{\partial x_k}a^i_k\frac{\partial \lambda_i}{\partial x_j} = \sum_{k} \frac{\partial v}{\partial x_k}\frac{\partial \sum_{i} a^i_k \lambda_i}{\partial x_j} = \sum_{k} \frac{\partial v}{\partial x_k}\frac{\partial x_k}{\partial x_j} = \frac{\partial v}{\partial x_j}.$$

We are now ready to conclude:

**Theorem 5.** Suppose $K$ is a triangle and $v \in C^2(K)$. Let $\pi v \in P_1(K)$ such that $\pi v(a^i) = v(a^i)$, i.e. $\pi v$ is the interpolation of $v$. Then, we have

$$\|v - \pi_h v\|_{L^\infty(K)} \leq 2h^2_K \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)}$$

$$\max_{|\alpha|=1} \|D^\alpha (v - \pi_h v)\|_{L^\infty(K)} \leq \frac{h^2_K}{\rho_K} \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)}$$

**Proof.** Let $\lambda_i$ be the basis of $P_1(K)$. Then, $\pi_h v = \sum_{i=1}^{3} v(a^i)\lambda_i$.

Now, by Taylor expansion, we have

$$v(a^i) = v(x) + \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x)(a^i_j - x_j) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 v(\xi)}{\partial x_j \partial x_k}(a^i_j - x_j)(a^i_k - x_k)$$

$$= v(x) + p_i(x) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 v(\xi)}{\partial x_j \partial x_k}(a^i_j - x_j)(a^i_k - x_k)$$
Denote the second order term by $R(x, a^i)$. Plugging this in and using Lemma 1, we have

$$
\pi_h v(x) = v(x) + \sum_{i=1}^{3} R(x, a^i) \lambda_i
$$

Clearly, $|R| \leq 2h^2 K \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty}$ since $|\lambda_i| \leq 1$.

For the derivative, one has

$$
\partial_l (\pi_h v) = \sum_i v(a^i) \frac{\partial \lambda_i}{\partial x_l}
$$

We now plug in the expansion for $v(a^i)$.

$$
\sum_i v(x) \frac{\partial \lambda_i}{\partial x_l} = v(x) \frac{\partial}{\partial x_l} \sum_i \lambda_i = 0.
$$

$$
\sum_i p_i(x) \frac{\partial \lambda_i}{\partial x_l} = \frac{\partial v}{\partial x_l}.
$$

Hence

$$
|\partial \pi_h^l v / \partial x_l - \partial v / \partial x_l| = \sum_i R(x, a^i) \frac{\partial \lambda_i}{\partial x_l}
$$

The estimate follows by the fact that $|\partial \lambda_i / \partial x_l| \leq 1/\rho_K$.