Math 660-Lecture 17: Spectral methods: Fourier pseudo-spectral method

Spectral accuracy of Fourier pseudo-spectral method

The spatial accuracy of Fourier pseudo-spectral method is usually $O(h^m)$ \(\forall m > 0\) for smooth functions and $O(c^N)$ for analytic functions. This is known as the spectral accuracy. We aim to understand why we have such spectral accuracy.

1 Aliasing formula

We now consider the effect of sampling. Suppose we discretize the space with step $h$ and $x_j = jh$. Let’s recall the semi-discrete Fourier Transform of $v$, given by

$$\hat{v}(\xi) = h \sum_{j=-\infty}^{\infty} v_j e^{-i\xi x_j}.$$ 

Note that this is different from the semi-discrete transform we have used in the von-Neumann analysis. The only difference is that we don’t have the extra $\frac{1}{\sqrt{2\pi}}$ here. This extra constant doesn’t matter.

**Theorem 1.** Suppose $u \in L^2(\mathbb{R})$ and has a first derivative with bounded variation. Let $v_j = u(x_j)$. $\hat{v}(\xi)$ is the semi-discrete Fourier transform of $v$ and $\hat{u}(\xi)$ is the Fourier Transform of $u$. Then, for any $\xi \in [-\pi/h,\pi/h)$, we have

$$\hat{v}(\xi) = \sum_{j=-\infty}^{\infty} \hat{u}(\xi + 2\pi j/h).$$

*Comment:* This aliasing formula is related to the Nyquist sampling theorem in signaling processing.

We now consider the aliasing formula for functions defined on the torus. Recall that the DFT is defined by

$$\hat{v}_k = \sum_{n=1}^{N} v_n e^{-ikx_n}$$

**Theorem 2.** Suppose $u$ is a good periodic function on $[0,2\pi)$ such that its Fourier series converges to $u$. Let $v_j = u(x_j)$. Then, the DFT of $v$ and
Fourier series of $u$ are related by

$$\hat{v}_k = N \sum_{m=-\infty}^{\infty} \hat{u}_{k+mN}, k = -N/2 + 1, \ldots, N/2.$$ 

Note that here we used a different scaling for DFT compared with semi-discrete Fourier Transform and the Fourier series for functions on the torus. That’s why we have the extra $N = 2\pi/h$ here.

For the functions defined on $\mathbb{R}$, we have

**Corollary 1.** With the same notations, we have the following claims for any $\xi \in [-\pi/h, \pi/h)$:

- If $u$ has $p-1$ continuous derivatives and are in $L^2$ for some $p \geq 1$ and a $p$th derivative of bounded variation, then
  $$|\hat{u}(\xi) - \hat{v}(\xi)| = O(h^{p+1}).$$

- If $u$ has infinitely many continuous derivatives in $L^2$, then
  $$|\hat{u}(\xi) - \hat{v}(\xi)| = O(h^m), \forall m > 0.$$

- If $u$ can be analytically extended to $|Im(z)| < a$ such that there exits a constant $C$ independent of $y \in (-a, a)$, $\|u(\cdot + iy)\|_{L^2} \leq C$, then
  $$|\hat{u}(\xi) - \hat{v}(\xi)| = O(e^{-\pi(a-\epsilon)/h}), \forall \epsilon > 0.$$

**Proof.** We can estimate that

$$|\hat{u}(\xi) - \hat{v}(\xi)| \leq \sum_{j \neq 0} |\hat{u}(\xi + 2\pi j/h)| \sim \frac{h}{2\pi} \int_{|\xi| > \pi/h} |\hat{u}(\xi)| d\xi$$

In the first case, we have $Ch \int_{|\xi| > \pi/h} \frac{1}{|\xi|^{p+1}} d\xi = O(h^{p+1})$. The other cases follow similarly.

For the periodic case, we then have the following corollary:

**Corollary 2.** Suppose $u \in C^{(p-1)}([0, 2\pi], \mathbb{C})$, $u^{(j)}(0) = u^{(j)}(2\pi)$ for $j = 0, 1, \ldots, p-1$ and that $u^{(p)}$ is piecewise continuous. Let $\hat{v}_k$ be the DFT of its sampling and $\hat{u}_k$ be its Fourier series coefficients. Then,

$$|\hat{u}(k) - \frac{1}{N} \hat{v}_k| = O(N^{-p}) = O(h^p).$$
2 The spectral accuracy

Using the aliasing formula, it’s not hard to find the spectral accuracy results.

For the semi-discrete Fourier transform, suppose \( \hat{w}(\xi) = (ik)^\nu \hat{v}(\xi) \) for \( \xi \in [-\pi/h, \pi/h) \) and \( v_j = u(x_j) \). Then, we have the following theorem:

**Theorem 3.** For any \( \xi \in [-\pi/h, \pi/h) \):

- If \( u \) has \( p-1 \) continuous derivatives and are in \( L^2 \) for some \( p \geq \nu + 1 \) and a \( p \)th derivative of bounded variation, then
  \[
  |w_j - u^{(\nu)}(x_j)| = O(h^{p-\nu}).
  \]
- If \( u \) has infinitely many continuous derivatives in \( L^2 \), then
  \[
  |w_j - u^{(\nu)}(x_j)| = O(h^m), \forall m > 0.
  \]
- If \( u \) can be analytically extended to \( |Im(z)| < a \) such that there exists a constant \( C \) independent of \( y \in (-a,a) \), \( \|u(\cdot + iy)\|_{L^2} \leq C \), then
  \[
  |w_j - u^{(\nu)}(x_j)| = O(e^{-\pi(a-\epsilon)/h}), \forall \epsilon > 0.
  \]

Similarly, we have

**Theorem 4.** Suppose \( u \in C^{(p-1)}([0,2\pi], \mathbb{C}) \), \( u^{(j)}(0) = u^{(j)}(2\pi) \) for \( j = 0, 1, \ldots, p - 1 \) and that \( u^{(p)} \) is piecewise continuous. Let \( \hat{v}_k \) be the DFT of its sampling and \( \hat{u}_k \) be the Fourier series coefficients. Suppose \( \hat{w} = (ik)^\nu \hat{v} \) and \( \nu \leq p - 1 \). Then,

\[
|w_j - u^{(\nu)}(x_j)| = O(h^{p-\nu}).
\]

The proof follows easily from the aliasing formulas. We would like to omit here.

3 Finite Element method for 1D second order elliptic equations

(In Sec. 1.1 of Johnson).

Consider the elliptic equation

\[-(a(x)u')' + c(x)u(x) = f(x), x \in (0,1), u(0) = u(1) = 0. \quad (D)\]

where \( a(x) \geq a > 0 \) and \( c(x) \geq 0 \).

- In the book, the problem is a special case, \(-u'' = f \). Since there’s no big difference from that equation, let’s use this more general form.
3.1 Minimization of Energy and variational form (weak formulation)

The equation corresponds to an energy

\[ F(u) = \int_0^1 \frac{1}{2} a(x) u'(x)^2 \, dx + \int_0^1 \frac{1}{2} cu^2 \, dx - \int_0^1 f(x) u(x) \, dx. \]

It’s easy to show using the variational principle or principle of virtue work that the elliptical equation is the problem

\[ \min_u F(u), \quad u(0) = u(1) = 0 \quad (M) \]

From this energy form, we take the variation and have

\[ a(u, v) = \int_0^1 a(x) u'(x) v'(x) \, dx + \int_0^1 cuv \, dx = \int_0^1 f(x) v(x) \, dx, \]

where \( v = \delta u \). This is the weak formulation of the differential equation. The variational problem can be derived easily formally: we multiply any test function \( v \) with \( v(0) = v(1) = 0 \) and integrate.

The variational problem is convenient since it can be used for equations like \(-(au')' + bu' + cu = f\), yielding

\[ \int_0^1 a(x) u'(x) v'(x) \, dx + \int_0^1 bu'v \, dx + \int_0^1 cuv \, dx - \int_0^1 f(x) v(x) \, dx = 0 \]

where \( c - \frac{1}{2} b' \geq 0 \) is required. Note that the second term can also be changed to \(-\int_0^1 bu'v' \, dx - \int_0^1 b'w vdx\) which doesn’t really matter.

Assume that \( a, b, c \) are continuous and \( a(x) \geq a > 0, c - \frac{1}{2} b' \geq 0 \). The weak form (or variation formulation) of \(-(au')' + bu' + cu = f\) is:

Find \( u \in H_0^1 \) such that

\[ \int_0^1 a(x) u'(x) v'(x) \, dx + \int_0^1 bu'v \, dx + \int_0^1 cuv \, dx - \int_0^1 f(x) v(x) \, dx = 0 \quad (V) \]

for all \( v \in H_0^1 \), where

\[ H_0^1 = \left\{ \int_0^1 (v')^2 + v^2 \, dx < \infty, v(0) = v(1) = 0 \right\} \]

is the space in which the weak formulation make sense.

In Mathematics, \( H_1^0 \) is a kind of Sobolev spaces (Read Sec. 1.5 for more details). The \( H^1 \) norm is defined to be

\[ \|v\|_{H^1} = \sqrt{\int_0^1 (v')^2 + v^2 \, dx}. \]
Remark 1. If the solutions are nice enough, then (D), (M) and (V) are equivalent. If the data are not smooth enough, (D) may not have classical solutions but (V) or (M) may still have solutions. These solutions will then be called the weak solutions of the PDE.