Math 660-Lecture 10: FDM for parabolic PDEs

1 Parabolic equations in multi-dimensional case

We’ll consider 2D heat equation for simplicity (3D will be similar):

\[ u_t = \Delta u = u_{xx} + u_{yy}. \]

1.1 MOL (method of lines) type schemes

Let’s use \( \Delta_h \) to mean the five-point difference for Laplacian.

**Forward Euler:**

\[ \frac{u_{j}^{n+1} - u_{j}^{n}}{k} = \Delta_h u_{j}. \]

The LTE is \( O(k + h^2 + h^2) \). Using von Neumann analysis (we should use \( u_{j}^{n} = e^{ix_j \xi_1 + iy_k \xi_2} \)), we find

\[ g(\xi) = 1 - 4 \frac{k}{h^2} (\sin^2(\xi_1 h/2) + \sin^2(\xi_2 h/2)). \]

Then, we need \( 4k/h^2 \leq 1 \) for this to be stable. This is more severe compared with the 1D case.

For the Crank-Nicolson,

\[ \frac{u_{j}^{n+1} - u_{j}^{n}}{k} = \frac{1}{2} [\Delta_h u_{j}^{n} + \Delta_h u_{j}^{n+1}]. \]

The LTE is \( O(k^2 + h^2 + h^2) \) and it is unconditionally stable. For this scheme, the issue arises for how to solve this linear system.

\[ (I - \frac{k}{2} \Delta_h)u_{j}^{n+1} = (I + \frac{k}{2} \Delta_h)u_{j}^{n}. \]

Here, the matrix for \( I - \frac{k}{2} \Delta_h \) is not tridiagonal and the direct method is not that fast. The matrix is sparse so we don’t want to use Gauss elimination (In Matlab, if you use backslash, and the matrix is not constructed in a sparse way, probably it will be solved by Gauss elimination). Hence, the iterative methods are desired. The condition number for this matrix is \( O(k/h^2) \), which is not very big. Using the initial guess \( (u^{n+1})^{(0)} = u^n \), the convergence can be obtained quickly.
1.2 Other optional methods

1.2.1 Locally one dimensional method (LOD)

From above we see that the Crank-Nicolson for 2D will produce a matrix that is not tridiagonal. The idea of LOD is to use a time splitting method. Form $t^n$ to $t^{n+1}$, the equation

$$u_t = u_{xx} + u_{yy},$$

is split into two steps

$$u_t = u_{xx},$$

$$u_t = u_{yy}$$

(The time splitting method is to split $u_t = Au + Bu$ into $u_t = Au$ and $u_t = Bu$. We’ll come to this later for more details.)

We then apply the Crank-Nicolson for both and obtain the LOD method:

$$\frac{u^*_ij - u^n_{ij}}{k} = \frac{1}{2}(D^2_x u^n_{ij} + D^2_x u^*_{ij}),$$

$$\frac{u^{n+1}_{ij} - u^*_{ij}}{k} = \frac{1}{2}(D^2_y u^*_{ij} + D^2_y u^{n+1}_{ij})$$

Then, for each step, we have tridiagonal matrices and we can invert them easily. Since we have two 1D Crank-Nicolson for each step, the method is unconditionally stable.

For this method, one must determine the boundary conditions for $u^*$ carefully. Note that $u^*$ is not a physical quantity and it is not $u^{n+1/2}$. Read P198 to understand how to impose boundary conditions for LOD.

1.2.2 Alternating Direction Implicit Methods (ADI)

In the LOD method, the intermediate quantity $u^*$ is not physical. Then, the following ADI method gives physical intermediate step quantities:

$$\frac{u^*_ij - u^n_{ij}}{k/2} = D^2_x u^*_{ij} + D^2_y u^n_{ij}$$

$$\frac{u^{n+1}_{ij} - u^*_{ij}}{k/2} = D^2_x u^*_{ij} + D^2_y u^{n+1}_{ij}$$

In the first $k/2$ time, we make $x$ implicit; in the second $k/2$ time, we make $y$ implicit. For each $k/2$ step, the local truncation error is $O(k + h^2 + h^2)$.
but the whole local truncation error is $O(k^2 + h^2 + h^2)$ because the errors in the two steps cancel.

To see this, we are going to eliminate $u^*$ and find the formula from $u^n$ to $u^{n+1}$ directly. We have

$$(1 + \frac{1}{4} \frac{k^2}{h^4} D_x^2 D_y^2) \frac{u_{ij}^{n+1} - u_{ij}^n}{k} = \frac{1}{h^2} (D_x^2 + D_y^2) \frac{u_{ij}^{n+1} + u_{ij}^n}{2}$$

The method is unconditionally stable. Compared with the LOD, the unconditional stability is not so obvious. Here, we use Von Neumann analysis to see this. The factor for the first half step is given by

$$g_1(\xi_1, \xi_2) \frac{1}{k/2} = \frac{1}{h^2} (g(\xi_1, \xi_2)(2 \cos(\xi_1 h) - 2) + 2 \cos(\xi_2 h) - 2)$$

This gives

$$g_1(\xi_1, \xi_2) = \frac{1 - 2 \sin^2(\xi_2 h/2) \frac{k}{h^2}}{1 + 2 \sin^2(\xi_1 h/2) \frac{k}{h^2}}$$

Similarly, we have $g_2$. Clearly, a single $g_i$ will not guarantee stability, but the product of them is

$$g = g_1 g_2 = \frac{(1 - 2 \sin^2(\xi_2 h/2) \frac{k}{h^2})(1 - 2 \sin^2(\xi_1 h/2) \frac{k}{h^2})}{(1 + 2 \sin^2(\xi_1 h/2) \frac{k}{h^2})(1 + 2 \sin^2(\xi_2 h/2) \frac{k}{h^2})}$$

We find

$$|g| \leq 1.$$  

The advantage of this method is that we have a series decouple linear systems whose matrices are tridiagonal.

For example, in the first $k/2$ step, we need to figure out $u^*$. Then, we can solve the values for a fixed $y_j$. Then, it is 1D and the matrix is tridiagonal.

**Code presentation:** Consider the diffusion equation

$$u_t = \Delta u, \quad (x, y) \in \Omega = [0, 1] \times [0, 1].$$

$$u_0(x, y) = \sin(2\pi x) \sin(2\pi y), \quad u = 0, \partial \Omega.$$  

Simulate the problem using ADI. We plot the error versus the spatial step $h$. Check how error changes with time step by yourself.