1 Upwind scheme for advection equation with variable coefficient

Consider the equation
\[ u_t + a(x)u_x = 0. \]

Applying the upwind scheme, we have
\[ \frac{u_j^{n+1} - u_j^n}{k} = -a_j \frac{1}{h} (u_j^n - u_{j-1}^n), \quad a_j \geq 0 \]
\[ \frac{u_j^{n+1} - u_j^n}{k} = -a_j \frac{1}{h} (u_j^n - u_{j+1}^n) \quad a_j < 0. \]

CFL condition is \( k\|a\|_\infty / h \leq 1. \)

The von Neumann analysis is not appropriate since the coefficients are not constant. The energy method for \( l^2 \) stability is so obvious. (There is some energy estimate for the leapfrog method.)

Here, we can do \( l^\infty \) analysis. The method can be written as
\[ u_j^{n+1} = (1 - k|a_j|)u_j^n + k|a_j| u_{j^*}^n, \quad j^* = j - 1, \quad a_j > 0 \text{ and } j^* = j + 1, \quad a_j < 0. \]

Hence,
\[ |u_j^{n+1}| \leq (1 - k|a_j|)\|u^n\|_\infty + k|a_j|\|u^n\|_\infty = \|u^n\|_\infty. \]

2 Modified equations: numerical dissipation and dispersion

Code presentation: In this example, we solve \( u_t + u_x = 0 \) numerically with initial data
\[ u_0(x) = \exp(-20(x - 2)^2) + \exp(-(x - 5)^2). \]

We’ll compare the upwind scheme and Lax-Wendroff scheme. To solve this problem, we truncate the domain to \([0, 20]\) up to time \( t = 10 \). Note that for the time interval we considered, \( u(0, t) \approx 0 \) (for better approximation, you can use \( u(0, t) = \exp(-20(t+2)^2) + \exp(-(t+5)^2) \)). For the right boundary, we use one-sided approximation for the finite difference.

It’s clear that the Lax-Wendroff is more accurate. If we look at the behavior of the schemes more closely, we find that the upwind scheme tends to smooth out the corners and it has some dissipating (diffusion) effect. The Lax-Wendroff however causes oscillation near the corners, which suggests that the Lax-Wendroff has the dispersion effect.
2.1 Analysis using modified equations

Compared with the original PDE, it’s possible to find a PDE that is better satisfied by our numerical method. These PDEs are called the modified equations.

- Consider the upwind scheme for $u_t + au_x = 0 \ a > 0$:

$$\frac{u_j^{n+1} - u_j^n}{k} = -\frac{a}{h}(u_j^n - u_{j-1}^n).$$

Suppose $v(x, t)$ is a smooth function that satisfies this numerical method exactly. Then, we have

$$\frac{v(x_j, t^{n+1}) - v(x_j, t^n)}{k} = -\frac{a}{h}(v(x_j, t^n) - v(x_{j-1}, t^n)).$$

$$\Rightarrow v_t(x_j, t^n) + \frac{1}{2}kv_{tt}(x_j, t^n) + O(k^2) = -a[v_x(x_j, t^n) - \frac{1}{2}v_{xx}h] + O(h^2)$$

Hence,

$$v_t + av_x = \frac{ah}{2}v_{xx} - \frac{1}{2}kv_{tt} + O(h^2 + k^2).$$

This suggests

$$v_{tt} = -av_{xt} + O(h + k) = a^2v_{xx} + O(h + k).$$

Inserting this into the term above, we have

$$v_t + av_x = \frac{a}{2}(h - ak)v_{xx} + O(h^2 + k^2)$$

This means that $v(x, t)$ satisfies the equation $v_t + av_x = \frac{a}{2}(h - ak)v_{xx}$ better than $v_t + av_x = 0$. The modified equation is advection-diffusion equation. Clearly, if $h - ak > 0$ or $\frac{ak}{h} < 1$, there is diffusion effect. This is called the numerical diffusion.

- Similarly, the modified equation for Lax-Wendroff is

$$v_t + av_x = -\frac{1}{6}ah^2(1 - (\frac{ak}{h})^2)v_{xxx} + O(k^3 + h^3)$$

The main error for the transport equation is $v_{xxx}$. If you compute the dispersion relation:

$$-i\omega + ai\xi = -\frac{1}{6}ah^2(1 - (\frac{ak}{h})^2)(-i)\xi^3 \Rightarrow \omega = a\xi - \frac{1}{6}ah^2(1 - (\frac{ak}{h})^2)\xi^3.$$ 

Hence, the main error term is dispersive and that is why the oscillation appears there.
Comment: The numerical dispersion relation for the FDM (or equivalently, the dispersion relation for the full modified equation) can be computed by inserting $u^n_j = \exp(i(\xi x_j - \omega t^n))$ into the finite difference method. If you compare this with von-Neumann, $e^{-i\omega(\xi)k} = g(\xi)$.

2.2 Adding numerical dissipation/diffusion

Sometimes, we desire to have numerical diffusion to smooth out numerical solutions and damp some useless modes.

Consider the Leapfrog method:

\[
\frac{u_j^{n+1} - u_j^{n-1}}{2k} = a \frac{u_{j+1}^n - u_{j-1}^n}{2h}.
\]

Let’s compute the exact numerical dispersion relation. Assuming $v(x, t) = \exp(i(\xi x - \omega t))$ satisfies this numerical method exactly. Then, we have

\[
\sin(\omega k) = \frac{ak}{h} \sin(\xi h).
\]

Then, we see that $\omega$ is a real function of $\xi$. Hence, for every order of $k$ and $h$, the Leapfrog method only has dispersion and there is no dissipation. That means all modes in the error for the advection equation will not damp.

Hence, if one wants to damp those modes, we can add numerical dissipation. One possible way is to solve the following FDM:

\[
\frac{u_j^{n+1} - u_j^{n-1}}{2k} = -a \frac{1}{2h}(u_{j+1}^n - u_{j-1}^n) - \epsilon h^4 \frac{1}{2k} \left( \frac{D^2}{4} \right)^2 u_j^{n-1} = 0.
\]

If you compute the numerical dispersion relation, you’ll see that the added term gives dissipation.

3 Second order hyperbolic equations: wave equation

Consider the wave equation

\[
u_{tt} = a^2 u_{xx},
\]

\[u(x, 0) = f(x), \quad u_t(x, 0) = g(x).\]
Note if we introduce $v = u_t, w = au_x$, we have the first order system of equations

\[
\begin{align*}
v_t - aw_x &= 0 \\
w_t - av_x &= 0.
\end{align*}
\]

Here, we use the simplest scheme:

\[
\frac{u_{j+1}^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} - a^2 \frac{u_{j+1}^{n+1} - 2u_j^n + u_{j-1}^n}{h^2} = 0.
\]

The boundary condition $u_t$ can be approximated by the ghost point method.

First of all, let’s consider the CFL condition. The PDE has two characteristic speeds. One is $dx/dt = a$ and one is $dx/dt = -a$. Then, the CFL condition implies that $a|k| \leq h$.

Let’s derive the accurate condition using von-Neumann analysis, which is $\frac{ak}{h} < 1$.

One can assume $u_j^n = g(\xi)^n e^{ix_j \xi}$ to get the amplification factor.

Here, let’s show the matrix formulation. Introducing

\[
\begin{align*}
v^n_j &= \frac{u_j^n - u_j^{n-1}}{k}, & w_{j+1/2}^n &= a \frac{u_{j+1}^n - u_j^n}{h}.
\end{align*}
\]

The scheme can be rewritten as

\[
\begin{align*}
\frac{v_j^{n+1} - v_j^n}{k} - a \frac{w_{j+1/2}^n - w_{j-1/2}^n}{h} &= 0 \\
\frac{w_{j-1/2}^n - w_{j-1/2}^{n-1}}{k} - a \frac{v_j^{n+1} - v_j^{n}}{h} &= 0.
\end{align*}
\]

By doing so, we have a two-time level scheme. Then, $v_j^n = v^n e^{ix_j \xi}$ and $w_j^n = w^n e^{ix_j \xi}$. Then, we have

\[
\begin{align*}
\frac{v_j^{n+1} - v_j^n}{k} - a \frac{w_{j+1/2}^n}{h} \sin(\frac{1}{2} \xi h) &= 0 \\
\frac{w_{j-1/2}^n - w_{j-1/2}^{n-1}}{k} - a \frac{v_j^{n+1} - v_j^{n}}{h} \sin(\frac{1}{2} \xi h) &= 0.
\end{align*}
\]

Letting $c = \frac{2ak}{h} \sin(\frac{1}{2} \xi h)$, we have

\[
\begin{align*}
v_j^{n+1} &= v_j^n + icw_j^n \\
w_{j-1/2}^{n+1} &= icv_{j+1/2}^n + (1 - c^2)w_{j-1/2}^n.
\end{align*}
\]
The e-vals satisfy $\mu^2 - (2 - c^2)\mu + 1 = 0$. We need $|\mu| \leq 1$. The product is 1 hence both eigenvalues must have magnitude 1. $(2-c^2) = \mu + \mu^{-1} = 2\cos(\theta)$. Hence, we need

$$c^2 \leq 4 \Rightarrow \frac{ak}{h} \leq 1.$$ 

However, the norm of the matrix may grow if the eigenvalues are repeated. This happens if $c = 0$ or $c^2 = 4$. $c = 0$ is fine since the matrix has two Jordan blocks. $c = \pm 2$ corresponds to $ak/h = 1$. This is bad since the matrix has only one Jordan block. The method is unstable. Hence, the stability condition is $ak/h < 1$ (unlike the advection equation).

## 4 The hyperbolic systems

- For 1D hyperbolic system $u_t + A u_x$ ($A$ is diagonalizable and has real e-values), the Lax-Friedrichs, Lax-Wendroff and Leapfrog schemes can be generalized easily. The stability condition should be satisfied for all eigenvalues.

- The non-trivial generalization is for upwind scheme and Beam-Warming. If eigenvalues are all nonnegative or all are nonpositive, the generalization is also easy. The issue arises if some are positive while some are negative. In this case, we should find the characteristic variables (i.e. the eigenvectors). For those with positive e-values, we use the $a > 0$ upwind and for those with negative e-vals, we use $a < 0$ upwind.

## 5 Nonlinear hyperbolic conservation laws & multi-dimension problems

The Finite Volume Methods are more suitable for nonlinear hyperbolic conservation laws. We’ll come back later. For multi-dimension problems, you can use LOD or ADI ideas to design schemes. We’ll not talk about them in our class. Those who are interested can read more references.