1 Advection equation

We consider the simplest hyperbolic equation \( u_t + au_x = 0 \).

1.1 First order schemes

1.1.1 The forward Euler

In Lecture 5, we have the following forward Euler time discretization method:

\[
\frac{u_j^{n+1} - u_j^n}{k} = -\frac{a}{2h}(u_{j+1}^n - u_{j-1}^n).
\]

This scheme is unstable for a fixed \( k/h \) ratio, but if \( k = O(h^2) \), we have convergence.

1.1.2 Lax-Friedrichs

To get a better method than the forward Euler, one may replace \( u_j^n \) with \( \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) \). Then, we have the Lax-Friedrichs scheme. It turns out that this has a better stability compared with the forward Euler.

\[
u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n).
\]

Taylor expansion shows that the local truncation error is \( O(k + h^2 + h^2/k) \).

von Neumann analysis gives the amplification factor:

\[
g(\xi) = \cos(\xi h) - \frac{ak}{h}i\sin(\xi h) \Rightarrow |g|^2 = 1 + \left(\frac{a^2k^2}{h^2} - 1\right)\sin^2(\xi h).
\]

Hence, if \( \frac{ak}{h} \leq 1 \), the method is \( (l^2) \) stable. Provided this is true, the method is also of first order accuracy.

One can also compute the eigenvalues to check the stability condition as done in the book.
1.1.3 Upwind schemes

Another way to improve the stability is to use one-sided finite difference for \( u_x \). There are then two options

\[
\frac{u_j^{n+1} - u_j^n}{h} = -a \frac{1}{h} (u_j^n - u_{j-1}^n)
\]

\[
\frac{u_j^{n+1} - u_j^n}{h} = -a \frac{1}{h} (u_{j+1}^n - u_j^n).
\]

Performing von-Neumann analysis, we obtain the amplification factor

\[
g(\xi) = 1 - \frac{a k}{h} (1 - e^{-ih\xi}) = (1 - \frac{a k}{h} (1 - \cos(\xi h))) - i \frac{a k}{h} \sin(h\xi)
\]

\[
g(\xi) = 1 - \frac{a k}{h} (e^{ih\xi} - 1) = (1 + \frac{a k}{h} (1 - \cos(\xi h))) - i \frac{a k}{h} \sin(h\xi))
\]

Now, it’s clear that the first scheme is stable if \( a > 0 \) and \( ak/h \leq 1 \). It is unstable if \( a < 0 \). For the second scheme, it is stable if \( a < 0 \) and \( ak/h \geq -1 \).

When \( a > 0 \), the information of the PDE moves to right. The information flowing to \( j \) is from \( j - 1 \). Hence, we use \( u_{j-1}^n \) and \( u_j^n \). When \( a < 0 \), we should use \( u_j^n \) and \( u_{j+1}^n \). The one-sided differences therefore follows the direction of the moving or direction of ‘wind’. Hence, they are called ‘upwind schemes’.

\textit{Comment:} If the stability condition (CFL condition, see below) is satisfied, the method is also \( l^\infty \) stable. As we’ll see below.

1.1.4 Understanding the stability conditions

We derived the stability conditions using von-Neumann analysis. Now, let’s understand them in other viewpoints.

• Diffusion coefficient.

The Lax-Friedrichs can be rewritten as

\[
\frac{u_j^{n+1} - u_j^n}{k} + aD_0u_j = \frac{h^2}{2k}\Delta_k u_j^n.
\]

The upwind scheme for \( a > 0 \) can be rewritten as

\[
\frac{u_j^{n+1} - u_j^n}{k} + aD_0u_j = \frac{ah}{2}\Delta_k u_j^n.
\]
For the upwind scheme, we see clearly that if $a < 0$, the coefficient for the diffusion is negative and such model is ill-posed and it must be unstable. Hence, this upwind scheme must be used for $a > 0$.

Further, the eigenvalues for

$$\frac{u_j^{n+1} - u_j^n}{k} + aD_0u_j = \epsilon \Delta_h u_j^n.$$  

are

$$\lambda_p = -\frac{ia}{h} \sin(2\pi ph) - \frac{2\epsilon}{h^2} (1 - \cos(2\pi ph)).$$

Hence, $k\lambda_p$ falls on the ellipse \(\left(\frac{x}{2\epsilon h^2} + 1\right)^2 + \frac{y^2}{(a^2 h^2/2)^2} = 1\). To ensure this ellipse to be in the stability region of the forward Euler, we must have \(|ak/h| \leq 1\) and \(\frac{2\epsilon k^2}{h^2} \leq 1\).

- **Courant-Friedrichs-Lewy(CFL) conditions**

**Condition 1.** A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as $k, h \to 0$.

We look at the upwind scheme and the forward Euler for heat equation.

For the upwind scheme for $a > 0$, $u_j^{n+1}$ depends on $u_j^n$ and $u_{j-1}^n$. For the PDE, $u(x_j, t^{n+1})$ depends on $u(x_j - ak, t^n)$. Hence, we need $x_j - ak$ to be between $x_{j-1}$ and $x_j$, or $ak \leq h$.

For the heat equation with forward Euler, $u_j^{n+1}$ depends on $u_{j-1}^n, u_j^n, u_{j+1}^n$. For the PDE, the dependence domain is the whole axis. However, if $k = O(h^2)$, as $h \to 0$, $x_{j-1}, x_j, x_{j+1}$ will be the whole axis as $h \to 0$ since $2h/k \to \infty$.

### 1.2 Second order schemes

The schemes above have first order accuracy. We now look at some second order schemes.

**1.2.1 Leapfrog and Crank-Nicolson**

If we modify the forward Euler and use the midpoint method for time and we have the leapfrog scheme:

$$\frac{u_j^{n+1} - u_j^n}{2k} = -\frac{a}{2h}(u_{j+1}^n - u_{j-1}^n).$$
The accuracy is $O(k^2 + h^2)$. The eigenvalues of the finite difference are $\lambda_p = -i \frac{a}{h} \sin(2\pi ph)$. Recall the stability region of the midpoint method is $(-i, i)$. Hence, if $|ak/h| < 1$, the method is stable. The von Neumann analysis will give the same result. In this case, we should assume $u^n_{j-1} = e^{ix_j \xi}$ and assume $u^n_j = g(\xi) e^{ix_j \xi}$, $u^{n+1}_j = g(\xi)^2 e^{ix_j \xi}$. The leapfrog involves three time levels.

The Crank-Nicolson is to apply trapezoidal in time. It is two-time level but it is implicit. The implicit schemes are not common in hyperbolic equations since hyperbolic equations are not stiff.

### 1.2.2 Lax-Wendroff

Consider

$$u(x_j, t^{n+1}) = u(x_j, t^n) + u_t(x_j, t^n)k + \frac{1}{2} u_{tt}(x_j, t^n)k^2 + O(k^3).$$

By the equation, $u_t = -au_x$ and $u_{tt} = -a^2u_{xx}$. Hence,

$$u(x_j, t^{n+1}) = u(x_j, t^n) - ak u_x + \frac{1}{2} a^2 k^2 u_{xx} + O(k^3).$$

We then obtain the following Lax-Wendroff method:

$$u^{n+1}_j = u^n_j - ak D_0 u^n_j + \frac{1}{2} a^2 k^2 D^2 u^n_j.$$

The local truncation error is $O(k^2 + h^2)$. Using von-Neumann or MOL eigenvalue approach, the stability condition is $|ak/h| \leq 1$.

### 1.2.3 Beam-Warming

In the Beam-Warming, the upwind idea is used. For $a > 0$, we use $u^n_{j-2}, u^n_{j-1}, u^n_j$ to approximate $u_x$ and $u_{xx}$. Then, we have

$$u^{n+1}_j = u^n_j - \frac{ak}{2h} (3u^n_j - 4u^n_{j-1} + u^n_{j-2}) + \frac{ak^2}{2h^2} (u^n_j - 2u^n_{j-1} + u^n_{j-2}).$$

Direct Taylor expansion shows that the local truncation error is $O(k^2 + h^2)$. The stability condition is better:

$$\frac{ak}{h} \leq 2.$$ 

You can use the von-Neumann, or CFL condition to derive this.