Math 660-Lecture 3: FDM for 2D elliptic equations

(Chap 3 in Leveque)

Some typical elliptic equations in 2D include

\[-a_1 u_{xx} - a_2 u_{xy} - a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = f, \quad a_2^2 - 4a_1a_3 < 0\]

and

\[-(\kappa u_x)_x - (\kappa u_y)_y = f, \quad \kappa > 0.\]

Note that we have an extra negative sign compared with the book. This makes no difference since we can redefine \(f\) but this convention is preferred in literature since the so-defined elliptic operator is positive.

1 Poisson equation

Consider the 2D poisson equation

\[-\Delta u = f, \quad \Omega = [0,1] \times [0,1], \]

\[u = g, \quad \partial \Omega.\]

1.1 5-point

A first way to approximate the Laplacian: 5 point stencil.

For simplicity, we use uniform step size for both directions:

\[\Delta x = \Delta y = h = 1/(m + 1).\]

\(u_{ij}\) represents the value at \(x = x_i = ih, y = y_j = jh\). We approximate \(\Delta_h = D_x^2 + D_y^2:\)

\[-\Delta_h u_{ij} = -\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} - \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} = f_{ij} = f(x_i, y_j).\]

This is called 5-point stencil since we only used five points.

To set up the matrix, we must order the points. The most straightforward way is to order them as follows:

\[u = (u_{11}, u_{21}, \ldots, u_{m1}, u_{12}, u_{22}, \ldots, u_{m2}, \ldots, u_{mm}).\]

This is convenient in Matlab since this actually corresponds to reshaping by columns of matrices. Other ordering may be possible to make the matrix even sparser.
How do we set up the matrix? The most convenient way is to introduce

\[ \mathbf{u}_i = (u_{1i}, u_{2i}, \ldots, u_{mi}). \]

Consider

\[ e = \text{ones}(m, 1); \]
\[ A_1 = \text{spdiags}([-2 \times e e], -1:1, m, m); \]

Then, we have

\[ \frac{1}{h^2} A_1 \mathbf{u}_i - \frac{1}{h^2} (\mathbf{u}_{i+1} - 2\mathbf{u}_i + \mathbf{u}_{i-1}) = \mathbf{f}_i \]

\[ f_{i1} = f(x_i, y_i) + \frac{1}{h^2} g(x_i, 0), \quad f_{im} = f(x_i, y_m) + \frac{1}{h^2} g(x_i, 1) \]

and \( f_{ij} = f(x_i, y_j) \) for others. From this equation, it’s clear that the big matrix \( M \) has \( m \times m \) blocks. The \( (p,p) \) block is \( -\frac{1}{h^2} (A_1 - 2I) \) and the \( (p,p-1) \) and \( (p,p+1) \) blocks are \( -\frac{1}{h^2} I \). Note that \( \mathbf{u}_0 \) and \( \mathbf{u}_{m+1} \) can be determined by the boundary values and can be moved to right hand side.

The big matrix can be constructed using

\[ A_1 = \text{spdiags}([\text{ones}(m, 1)] * [1, -2, 1], -1:1, m, m); \]
\[ M = -\text{kron}(A_1, \text{speye}(m)) + \text{kron}(\text{speye}(m), A_1) / h^2; \]

**Code presentation** Consider the case \( g = 0 \) and \( f = 2\pi^2 \sin(\pi x) \sin(\pi y). \)

### 1.1.1 Analysis of the scheme

**Consistency:** The LTE is defined to be

\[ \tau_{ij} = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{h^2} - f(x_i, y_j), \]

which measures how the true solution satisfies the numerical approximation. Direct Taylor expansion shows that \( \tau_{ij} = O(h^2) \). As before, consistency means \( |\tau| \to 0 \) as \( h \to 0 \).

**Stability:** We show the \( l^\infty \)-stability (not in the book). The goal is then to show that there exists \( C \) independent of \( h \) such that

\[ \|u\|_{\infty} \leq C(\|f\|_{\infty} + \|g\|_{\infty}). \]

To prove this, we first show the **discrete maximum principle**:

**Theorem 1.** Let \( \Omega_h \) be the set of all interior points, i.e. \( \Omega_h = \{ (x_i, y_j) \} \setminus \partial\Omega \).

Let \( \Gamma_h \) be the sample points on \( \partial\Omega \).

Suppose \( \Delta^2_h u_{i,j} \geq 0 \) for all \( (x_i, y_j) \in \Omega_h \). Then \( \max_{\Omega_h} u \leq \max_{\Gamma_h} u \).

Further, if \( \max_{\Omega_h} u = \max_{\Gamma_h} u \), then \( u \) is a constant.
Proof. The first condition implies

\[ u_{ij} \leq \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}). \]

Suppose \( \max_{\Omega_h} u \geq \max_{\Gamma_h} u \) and the maximum is achieved at \((x_i^*, y_j^*)\) which is inside. Since it is the maximum, by the inequality, they must be equal. Its neighbors are maximums as well if they are interior points. The same argument applies. Then, the values at all interior points and their neighbors are equal. This means \( u \) is a constant. Hence, the bigger sign never holds and if the equal sign holds, \( u \) is a constant. \( \square \)

**Theorem 2.** The 5-point stencil has a unique solution for any \( f, g \) and it is \( l^\infty \)-stable.

Proof. For the uniqueness, suppose there are two solutions \( u_1 \) and \( u_2 \). Then, \( \Delta_h^2(u_1 - u_2) = 0 \) and the boundary values of \( u_1 - u_2 \) are zero. The discrete maximum principle implies that \( u_1 - u_2 \leq 0 \) for all interior points. Then, switching the roles of \( u_1 \) and \( u_2 \), we have \( u_2 - u_1 \leq 0 \). Hence, \( u_1 = u_2 \).

For the \( l^\infty \) stability, consider an auxiliary function \( \phi \) such that \( \Delta_h^2 \phi = 1 \). Then,

\[ \Delta_h(u + \phi \|f\|_\infty) = -f + \|f\|_\infty \geq 0. \]

The discrete maximum principle implies that

\[ u + \phi \|f\|_\infty \leq \max_{\Gamma_h}(g + \phi \|f\|_\infty) \Rightarrow u \leq \|g\|_\infty + 2\|\phi\|_\infty \|f\|_\infty. \]

Then, one applies the same argument for \(-u\). The claims follows. To finish the proof, we must show that \( \phi \) exists. One example is

\[ \phi = \frac{1}{4}((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2). \]

\( \square \)

**Corollary 1.** \( \|E\|_\infty = \|u - \hat{u}\|_\infty \to 0 \) as \( h \to 0 \) where \( \hat{u} \) consists of the true values at the grid points.

For the \( l^2 \) convergence, as the 1D case, we should show the \( l^2 \) stability. On Page 63 in Leveque’s book, the eigenvalues are computed directly.

Here we provide another proof, which is interesting itself. Consider the case \( g = 0 \) for simplicity. \( g \neq 0 \) case can be shown as well.
Theorem 3. Consider \( g = 0 \), and define \( \|w\|_2 = \sqrt{h^2 \sum_{i,j} |w_{ij}|^2} \). Then,

\[
\|u\|_2 \leq \frac{1}{2} \|\Delta_h u\|_2 = \frac{1}{2} \|f\|_2.
\]

In other words, the 5-point stencil is \( l^2 \) stable.

The proof follows from the following two lemmas. The first is the discrete Poincare inequality:

Lemma 1. If \( g = 0 \), then

\[
\|u\|_2 \leq \frac{1}{2}(\|D_{+,x} u\|_2 + \|D_{+,y} u\|_2)
\]

where \( D_{+,x} u_{ij} = (u_{i+1,j} - u_{ij})/h \) and \( D_{+,x} u_{m+1,j} = 0 \). \( D_{+,y} \) is defined similarly.

The second is the discrete Green’s identity:

Lemma 2. Suppose both \( v \) and \( w \) vanish on the boundary, then

\[
-\langle \Delta_h v, w \rangle = \langle D_{+,x} v, D_{+,x} w \rangle + \langle D_{+,y} v, D_{+,y} w \rangle,
\]

where

\[
\langle v, w \rangle = h^2 \sum_{i=0}^{m+1} \sum_{j=0}^{m+1} v_{ij} w_{ij}.
\]

Proof. For the Poincare:

\[
|u_{ij}|^2 = \left| \sum_{p=i}^{m} (u_{p+1,j} - u_{pj}) \right|^2 = \left( \sum_{p=i}^{m} h |D_{+,x} u_{pj}| \right)^2 \leq \left( \sum_{p=0}^{m} h |D_{+,x} u_{pj}|^2 \right) \left( \sum_{p=0}^{m} h \right) \leq \sum_{p=0}^{m} h |D_{+,x} u_{pj}|^2.
\]

Hence,

\[
\|u\|_2^2 = \sum_{i,j} h^2 |u_{ij}|^2 \leq \sum_{i,j} h^3 \sum_{p=0}^{m} |D_{+,x} u_{pj}|^2 = \sum_{p,j} h^2 |D_{+,x} u_{pj}|^2.
\]

or

\[
\|u\|_2 \leq \|D_{+,x} u\|_2.
\]
Similarly,
\[ \|u\|_2 \leq \|D_{+y}u\|_2. \]

Adding these two yields the desired inequality.

For the discrete Green’s identity:

\[ -\langle \Delta_h v, w \rangle = -\sum_{i=0}^{m+1} \sum_{j=0}^{m} \Delta_h v_{ij} w_{ij} = -\sum_{i=1}^{m} \sum_{j=1}^{m} \Delta_h v_{ij} w_{ij} \]

\[ = \sum_{i=1}^{m} \sum_{j=1}^{m} (2v_{ij} - v_{i+1,j} - v_{i-1,j}) w_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{m} (2v_{ij} - v_{i,j+1} - v_{i,j-1}) w_{ij}. \]

The first term equals

\[ \sum_{i=1}^{m} \sum_{j=1}^{m} (v_{ij} - v_{i-1,j}) w_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{m} (v_{i+1,j} - v_{i,j}) w_{ij} \]

\[ = \sum_{i=0}^{m} \sum_{j=1}^{m} (v_{i+1,j} - v_{ij}) w_{i+1,j} - \sum_{i=1}^{m} \sum_{j=0}^{m} (v_{i+1,j} - v_{ij}) w_{ij} \]

\[ = \sum_{i=0}^{m} \sum_{j=0}^{m} (v_{i+1,j} - v_{ij}) w_{i+1,j} - \sum_{i=0}^{m} \sum_{j=0}^{m} (v_{i+1,j} - v_{ij}) w_{ij} \]

\[ = \sum_{i=0}^{m} \sum_{j=0}^{m} h^2 D_{+x} v_{ij} D_{+x} w_{ij} = \langle D_{+x} v, D_{+x} w \rangle. \]

The second term is similarly computed. Hence, the claim discrete Green’s identity holds.

The theorem is easy to prove now using these two lemmas:

\[ \|u\|_2 \leq \frac{1}{4} (\|D_{+x}u\|_2 + \|D_{+y}u\|_2)^2 \leq \frac{1}{2} (\|D_{+x}u\|^2 + \|D_{+y}u\|^2) = \frac{1}{2} (-\Delta u, u) \leq \frac{1}{2} \|\Delta_h u\|_2 \|u\|_2. \]

**Corollary 2.** \( \|E\|_2 = \|u - \hat{u}\|_2 \to 0 \) as \( h \to 0 \) where \( \hat{u} \) consists of the true values at the grid points.

### 1.2 9-point scheme

Instead of using the 5-points, one can use the points nearby and obtain:

\[ \Delta_9 u_{ij} = \frac{1}{6h^2} (4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} + u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 20u_{ij}). \]
Note that this is again a second order accurate approximation for the Laplacian. However, if we modify $f$ with

$$\tilde{f}_{ij} = f_{ij} - \frac{h^2}{12} \Delta_h f_{ij},$$

the accuracy is $O(h^4)$. (Note that we have a sign difference from the book because we consider $-\Delta u = f$ instead of $\Delta u = f$.)

Exercise: Think about constructing the matrix in Matlab.

2 Other elliptic equations

Read Page 66 in Leveque’s book. We choose to omit. The essential ideas are the same.