Math 660-Lecture 10: FDM for mixed type equations

1 Time splitting method (fractional step method)

1.1 The basic splitting

Consider the equation

\[ u_t = A(u) + B(u), \]

where \( A, B \) are two Markovian operators. The idea of the fractional step method is to solve the following two equations from \( t^n \) to \( t^{n+1} \):

\[ u_t = A(u) \Rightarrow u^n \rightarrow u^* \]
\[ u_t = B(u) \Rightarrow u^* \rightarrow u^{n+1}. \]

**Theorem 1.** If \( A \) and \( B \) are linear bounded, constant operators, then the splitting error is of first order. If \( A \) and \( B \) commute, then there’s no splitting error.

The argument is easy to make. By the methods, we have

\[ u^* = e^{kA} u^n, \quad u^{n+1} = e^{kB} u^* = e^{kB} e^{kA} u^n. \]

For the original equation, we should have

\[ u(t^{n+1}) = e^{k(A+B)} u(t^n). \]

By the definition of exponential of operators, we have

\[ e^{k(A+B)} = I + k(A + B) + \frac{1}{2} k^2 (A + B)^2 + O(k^3) \]
\[ = I + k(A + B) + \frac{1}{2} k^2 (A^2 + AB + BA + B^2) + O(k^3) \]

Similarly,

\[ e^{kA} = I + kA + \frac{1}{2} k^2 A^2 + O(k^3) \]
\[ e^{kB} = I + kB + \frac{1}{2} k^2 B^2 + O(k^3). \]
Hence,

\[ e^{kB}e^{kA} = (I + kB + \frac{1}{2}k^2B^2 + O(k^3))(I + kA + \frac{1}{2}k^2A^2 + O(k^3)) \]

\[ = I + kA + kB + \frac{1}{2}k^2A^2 + \frac{1}{2}k^2B^2 + k^2BA + O(k^3) \]

Assuming that \( u^n = u(t^n) \), we find the local truncation error is

\[ LTE = \frac{1}{k}(u^{n+1} - u(t^n)) = \frac{1}{k} \left( I + kA + kB + \frac{1}{2}k^2A^2 + \frac{1}{2}k^2B^2 + k^2BA + O(k^3) \right. \]

\[ \left. -(I + k(A + B) + \frac{1}{2}k^2(A^2 + AB + BA + B^2) + O(k^3)) \right) u^n = \frac{1}{2}k(BA - AB)u^n + O(k^2). \]

Splitting method decouples the problem into two problems and each may be solved easily. Also, splitting makes the numerical method much easier and we may have less constraints for the stability requirement.

A disadvantage of fractional step method is that we may have issue for the boundary conditions of \( u^* \). If the boundary conditions for \( u \) is 0 for all time \( t = 0 \), then we may apply the same boundary condition for \( u^* \). For other cases, we should treat this problem carefully, as \( u^* \) is not a physical quantity.

These features can be seen by the LOD method for the 2D heat equation.

1.2 One example

Consider the advection-reaction equation:

\[ u_t + a(x)u_x = -\lambda u, \quad a(x) \geq 0. \]

If we don’t split, we probably want to do

\[ \frac{u_j^{n+1} - u_j^n}{k} + a_j \frac{u_j^n - u_{j-1}^n}{h} = -\lambda u_j^{n+1}. \]

If we do splitting, then we have:

- Problem A: \( u_t + a(x)u_x = 0 \).
- Problem B: \( u_t = -\lambda u \).
The first can be solved using the upwind scheme while the second can be solved exactly:

\[
\frac{u_j^* - u_j^n}{k} = -a_j \frac{u_j^n - u_{j-1}^n}{h},
\]

\[
u_{j+1}^n = \exp(-k\lambda)u_j^*.
\]

Another example is the advection diffusion equation: \(u_t + au_x = \nu u_{xx}\). Previously, we have seen the unsplit methods. If we use the splitting method \(u_t + au_x = 0\), and \(u_t = \nu u_{xx}\). The first one can be solved using the upwind scheme while the second one can be solved using Crank-Nicolson method.

1.3 Strang splitting

To improve the accuracy of the splitting method, we may use the following Strang splitting:

\[
\begin{align*}
u_t &= A(u), \text{ for time } k/2, \\
u_t &= B(u), \text{ for time } k, \\
u_t &= A(u), \text{ for time } k/2.
\end{align*}
\]

By direct Taylor expansion, one can show that the local truncation error for this method is \(O(k^2)\).

In real implementation, we don’t solve the three steps for one time interval with length \(k\). The observation

\[
(\exp(\frac{1}{2}kA) \exp(kB) \exp(\frac{1}{2}kA))^n = \exp(\frac{1}{2}kA) \exp(kB)(\exp(kA) \exp(kB))^{n-1} \exp(\frac{1}{2}kA),
\]

allows us to solve two steps essentially for one time interval.

1.4 The projection method for Navier-Stokes equations

Consider the viscous incompressible Newtonian fluid, which can be described by the Navier-Stokes equations

\[
\begin{align*}
\rho(u_t + u \cdot \nabla u) - \mu \Delta u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{align*}
\]

We scale the lengths by \(L\) (the typical length), time by \(T\) and velocity by \(U = L/T\). The body force is correspondingly scaled by \(\rho L/T^2\). The
Reynolds number in this case is $Re = \frac{\rho U L}{\mu} = \frac{\rho L^2}{(\mu T)}$. The Navier-Stokes equations are then

$$u_t + u \cdot \nabla u + \nabla p = \frac{1}{Re} \Delta u + f$$

$$\nabla \cdot u = 0$$

Let’s assume that we have the no-slip boundary condition $u|_{\partial \Omega} = 0$.

For 2D problem, we can introduce the stream function and the problem can be solved. For 3D, the stream function formulation is not suitable.

The projection methods are fractional step methods. The idea is to consider

$$u_t + u \cdot \nabla u = \frac{1}{Re} \Delta u + f$$

$$u_t = -\nabla \psi$$

where $\psi$ is some approximation of the real pressure. Of course, we don’t know what $\psi$ is.

Hence, we can have the following discretization in time:

$$\frac{u^* - u^n}{k} + \frac{3}{2} u^n \cdot \nabla u^n - \frac{1}{2} u^{n-1} \cdot \nabla u^{n-1} = \frac{1}{2Re} \Delta (u^* + u^n) + f^{n+1/2}$$

$$\frac{u^{n+1} - u^*}{k} = -\nabla \psi^{n+1}$$

In the first equation, the advection term is discretized by the Adams-Bashforth while the diffusion term is discrized by Crank-Nicolson. For the second equation, we don’t know what $\psi$ is. However, we note that $u^{n+1}$ is divergence free. Then, $u^{n+1}$ and $k\nabla \psi^{n+1}$ are just Helmholtz decomposition of $u^*$, namely, any smooth vector field can be decomposed into a curl free field and a divergence free field and the decomposition is unique. Hence, for the second equation, we take the divergence on both sides and have:

$$\Delta \psi^{n+1} = \frac{1}{k} \nabla \cdot u^*.$$ 

Usually, the boundary condition of $\psi$ is the Neumann boundary condition $\partial_n \psi = 0$. We can apply the 5-point method for this elliptical problem. However, the obtained matrix is singular due to the Neumann boundary condition. We usually use some type of iterative methods to solve this problem. (There are conjugate gradient methods designed for this particular semidefinite problem.) As long as we have $\psi$, we can then find $u^{n+1}$. 

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For spatial discretization, the methods are straightforward. The first equation is an advection-diffusion equation. The advection term can be discretized by upwind methods while others can be discretized by centered difference.

Another issue is how to impose boundary conditions for $\mathbf{u}^*$ and $\psi$. This issue is not trivial. Those who are interested can read the papers by Kim&Moin, E&Liu.