Math 660-Lecture 1

1 Preliminaries

1.1 Classification of second order quasilinear PDEs

Consider the second order quasilinear PDEs for $u: \mathbb{R}^n \to \mathbb{R}$:

$$\sum_{i,j} a_{ij}(u, Du, x) \partial_{ij}^2 u + F(x, u, Du) = 0.$$ 

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a_{ij} = a_{ji}$.

$A = [a_{ij}]$. Just like quadratic forms, if $\det(A) = 0$, and the PDE can’t be reduced to one with a lower dimension, then the equation is said to be parabolic; If $A$ is definite, the equation is said to be elliptic; If $A$ is indefinite, the equation is said to be hyperbolic.

**Examples**

- The equation $Lu = -\sum_{i=1,j=1}^{n} a_{ij} \partial_{ij}^2 u + \sum_{i} b_{i} \partial_{i} u + cu = f$
  
  $(\sum_{ij} a_{ij} \xi_i \xi_j \geq \theta |\xi|^2)$ is elliptic. Typical example is the Poisson equation $\Delta u = f$.

- Letting $x_n = t$, the equation $u_t + Lu = f$ where $Lu = -\sum_{i=1,j=1}^{n-1} a_{ij} \partial_{ij}^2 u$
  
  and $\sum_{i=1,j=1}^{n-1} a_{ij} \xi_i \xi_j \geq \theta |\xi|^2$ is parabolic. Typical example is the heat equation $u_t = a^2 \Delta u$.

- The wave equation $u_{tt} - a^2 u_{xx} = 0$ is hyperbolic.

**From here on, we only talk about hyperbolic equations where $A$ only has one positive e-value or only one negative e-value.**

Parabolic and Hyperbolic equations are ‘evolutionary’. This is because they are causal: the state at $t = t_2$ (let’s say $t = x_n$ without loss of generality) has nothing to do with the states at $t > t_2$. Hyperbolic equations have characteristics, i.e. the curves for propagation of information. The solutions can be determined locally. The parabolic equations usually have infinite speed of propagation.

For elliptic equations, there is no concept of evolution or propagation, the solution is determined all at once by the boundary conditions.
1.2 Hyperbolic first-order quasi-linear system of equations

Now, let $u : \mathbb{R}^n \to \mathbb{R}^m$. Consider the system

$$u_t + \sum_{j=1}^{n} B_j(x,u)u_{x_j} = f$$

where $B_j$ is $m \times m$.

The system is said to be hyperbolic if for any linear combination of $B_j$, i.e. $B = \sum_j y_j B_j$ has $m$ real eigenvalues and is diagonalizable.

**Example:** the convection equation $u_t + au_x = 0$ is hyperbolic.

Usually, some hyperbolic second order equations can be reduced to first-order hyperbolic system. For $u_{tt} - a^2 u_{xx} = 0$, if we introduce $v = u_t + au_x$, then $v_t - av_x = 0$.

**Remark 1.** In the one-dimensional space case, the system $u_t + B(x,u)u_x = f$ is said to be elliptical if $B$ has no real eigenvalues.

An important class of hyperbolic systems is the conservation law $u_t + f(u)_x = 0$. $f(u)$ is called the flux function.

Consider the one equation case: $u$ can be understood as the density of the material. $f(u)_x$ is the divergence and it describes how the material flows. Integrating on $x$,

$$\frac{d}{dt} \int_{x_1}^{x_2} u \, dx = f(u)|_{x_2} - f(u)|_{x_1}.$$ 

If we integrate on the whole axis, we see $\int u \, dx = \text{const}$.

If there is source due to reaction, the equation will be modified to $u_t + f(u)_x = \psi(x,u)$.

1.3 Dispersion relation for linear equations

For linear evolution equations (parabolic and hyperbolic),

$$u_t = Lu,$$

where $L$ does not depend on $x$, $e^{i\xi \cdot x}$ is an eigenfunction of $L$. Apply the Fourier transform: $\hat{u} = \int u(x)e^{-i\xi \cdot x} \, dx$. Then, we have

$$\hat{u}_t = -i\omega(\xi)\hat{u}.$$
Then, $\hat{u} = c(\xi) \exp(-i\omega(\xi)t)$ and
\[ u = \frac{1}{2\pi} \int c(\xi)e^{-i\omega(\xi)t} e^{i\xi x} d\xi. \]

The solution is the superposition of different Fourier modes. The properties of the solutions are determined by $\omega(\xi)$ which is called the dispersion relation. In practice, we simply plug in $e^{i(\xi x - \omega t)}$ to find the dispersion relation.

If $\omega$ is real, the amplitude of each mode doesn’t decay and different mode has different speed, the equation is called dispersive. The equation is hyperbolic. If $\omega$ is not real and imaginary part is negative, the amplitude decays. The equation is called dissipating.

**Example:** $u_t = a^2 u_{xx}$: $-i\omega = a^2(-\xi^2)$ and $\omega = -ia^2\xi^2$.

**Example:** $u_{tt} = a^2 u_{xx}$: $-\omega^2 = a^2(-\xi^2)$ and $\omega = \pm a\xi$. Dispersive, wave-like.

**Example:** Schrodinger equation: $iu_t = -u_{xx}$. $\omega = \xi^2$. Schrodinger equation is wave-like and shares properties with hyperbolic equations.

- **What if the equation is nonlinear?** We can’t find the dispersion relation. However, if we have a stationary solution, we can linearize the equation around the stationary solution and then do the linear stability analysis.

- **What if L depends on x?** $e^{ix\xi}$ is not an eigenfunction, but we can fix $x$ and see the local behavior there.

## 2 Finite difference approximations

(Chap. 1 in Leveque.)

Consider that $h$ is a small step size.

The one sided differences:
\[ u'(x) \approx \frac{u(x + h) - u(x)}{h} = D_+ u(x), \quad u'(x) \approx \frac{u(x) - u(x - h)}{h} = D_- u(x) \]

Centered differences:
\[ u'(x) \approx \frac{u(x + h) - u(x - h)}{2h} = D_0 u(x) = \frac{1}{2} (D_+ + D_-) u(x), \]
\[ u''(x) \approx \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} = D^2 u(x) = D_+D_- u(x) \]
2.1 Local Truncation error

The so-called truncation error is $TE = RHS - LHS$. To estimate the truncation error, we do Taylor expansion and assume the solutions are nice:

$$u(x + h) = \sum_{n=0}^{\infty} \frac{h^n u^{(n)}(x)}{n!} = e^{h \frac{d}{dx}} u(x)$$

By Taylor expansion, it’s clear to see that the one-sided difference is first order accuracy ($TE = O(h)$) and the centered difference is of second order accuracy ($TE = O(h^2)$).

For example, $\frac{1}{h} [e^{h \frac{d}{dx}} - 1] u(x) = \frac{1}{h} (hu'(x) + \frac{1}{2} h^2 u''(x) + \ldots) = u'(x) + O(h)$

**Sample Code Presentation:** Comparison of the three different finite differences.

2.2 Deriving finite difference approximations

We consider the following problem:
Given $x_1, x_2, x_3$, $x_2 - x_1 = h_1$, $x_3 - x_2 = h_2$. Find $a, b, c$ such that

$$au(x_1) + bu(x_2) + cu(x_3) \approx u'(x_2)$$

is accurate as possible.

- Taylor expansion

$$[ae^{-h_1 \frac{d}{dx}} + b + ce^{h_2 \frac{d}{dx}}] u(x_2)$$

Using the expansion for exponentials, we need

$$a + b + c = 0$$

$$-h_1 a + ch_2 = 1$$

$$\frac{1}{2} ah_1^2 + \frac{1}{2} ch_2^2 = 0$$

Solving yields $a = -\frac{h_2}{h_1(h_1+h_2)}$, $b = \frac{h_2-h_1}{h_1h_2}$, $c = \frac{h_1}{(h_1+h_2)h_2}$. The error is $O(ah_1^3 + ch_2^3) = O(h^2)$.

- Interpolation. We use polynomial interpolation here. Different methods will yield the same polynomial. Here we choose Lagrange interpolation.

$$p(x) = u(x_1) \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + u(x_2) \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + u(x_3) \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$
Then,

\[ u'(x_2) \approx p'(x_2) = u(x_1) \frac{x_2 - x_3}{(x_1 - x_2)(x_1 - x_3)} \]

\[ + u(x_2) \frac{x_2 - x_3 + x_2 - x_1}{(x_2 - x_1)(x_2 - x_3)} + u(x_3) \frac{x_2 - x_1}{(x_3 - x_1)(x_3 - x_2)} \]

\[ = u(x_1) \frac{-h_2}{h_1(h_1 + h_2)} + u(x_2) \frac{h_2 - h_1}{h_1h_2} + u(x_3) \frac{h_1}{(h_1 + h_2)h_2} \]