1 Basics about Fluids.

By fluids we mean gases or liquids. They have properties in common (in the relation between stress and strain) which make them different from solids. We use a continuum model; that is, a fluid is made of a continuum of particles occupying a region of space, $\mathbb{R}^3$ or $\mathbb{R}^2$. (The continuum model can be obtained as a limit of more primitive models with molecules, but that is a long story we won’t touch on here.) To be definite, will assume we are in $\mathbb{R}^3$ for now, since that’s where we live.

Let $v(x, t) \in \mathbb{R}^3$ be the velocity of a fluid particle located at $x \in \mathbb{R}^3$ at time $t$. We think of a particle as physical material of infinitesimal size. Imagine that we color a particle red and watch it move as $t$ increases. Suppose $x_0$ denotes the initial position (at time $t = 0$) of some particle. Then its location $x$ at later time is some function $\Phi(x_0, t)$, the flow map. Since the particle follows the fluid motion,

$$\frac{d\Phi}{dt} = v(\Phi, t), \quad \Phi(x_0, 0) = x_0. \quad (1.1)$$

or $dx/dt = v(x, t)$ with initial condition $x = 0$ at $t = 0$. If we presume we (somehow) know $v$, this is an ordinary differential equation with initial condition. We often say the coordinates of $x$ are the Eulerian coordinates and those of $x_0$ are the Lagrangian coordinates. This terminology has no merit whatsoever, but since it is widely used I feel I have to perpetuate it.

Suppose $f$ is any scalar function of $(x, t)$. By composition with $\Phi$ we can regard it as a function of $(x_0, t)$. From the chain rule we get

$$\frac{df}{dt}(\Phi(x_0, t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_j} \frac{d\Phi_j}{dt} = \frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \quad (1.2)$$

where a sum over $j = 1, 2, 3$ is implicit. We can interpret this as the time derivative of $f$ along a particle path (that is, $x_0$ fixed rather than $x$). Such an expression occurs often, and we use the special notation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \quad (1.3)$$

as an equation of functions of $(x, t)$, without direct reference to $x_0$. $D/Dt$ is called the material derivative or substantial derivative. We also use the material derivative applied to vector quantities. The acceleration of a fluid particle is

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}. \quad (1.4)$$

This is a vector equation; the last term is in principle ambiguous, but we always mean the $i$th component is $\sum_j v_j (\partial v_i / \partial x_j)$. Think of this as the directional derivative of the vector field $v$ in the direction $v$. Sometimes we write $(v \cdot \nabla) v$, but I usually won’t bother to do that.
The main equation of motion for incompressible fluids comes from conservation of momentum. Assume for the moment that the fluid density $\rho(x, t)$ (mass per unit volume) is constant. The rate of change of momentum of a small bit of material in the fluid is given by the sum of forces acting on it; this is a form of Newton’s second law of motion. The rate of change of momentum per unit volume (assuming $\rho$ constant) is $\rho Dv/Dt$. One force (per unit volume) is $-\nabla p$, the gradient of pressure; the fluid accelerates toward a region of lower pressure. Another force is viscosity with the form $\mu \Delta v$, with $\mu$ constant, at least for a “Newtonian” fluid such as air or water. Thus we are led to the p.d.e.

$$\rho(v_t + v \cdot \nabla v) + \nabla p = \mu \Delta v.$$  \(1.5\)

We often divide through by $\rho$. If $\rho$ is constant, we use the kinematic viscosity $\nu = \mu/\rho$ in the last term and write

$$v_t + v \cdot \nabla v + \frac{1}{\rho} \nabla p = \nu \Delta v.$$  \(1.6\)

The above appears as an evolution equation for the velocity $v$ with $p$ as an extra unknown. It is complemented by the incompressibility condition

$$\nabla \cdot v = 0.$$  \(1.7\)

This is equivalent to the assumption that volume is conserved locally by the flow. To see this equivalence, we note that the Jacobian $J(x_0, t) = \det \nabla \Phi(x_0, t) = \det \partial \Phi/\partial x_0$ obeys

$$\frac{\partial J(x_0, t)}{\partial t} = (\nabla \cdot v)J(x_0, t), \quad v = v(\Phi(x_0, t), t)$$  \(1.8\)

as shown in many textbooks. Thus $J \equiv 1$ is the same as $\nabla \cdot v \equiv 0$.

The momentum equation actually holds even if $\rho$ is variable; this fact depends on the conservation of mass, not discussed here, and the incompressibility. More thorough derivations of the equations are given in textbooks such as the one by Chorin and Marsden. Liquids are incompressible under usual circumstances. Flow of a gas may be incompressible or not; generally it is if the velocity is not too large. (More precisely, a gas is incompressible in the limit that the Mach number approaches zero.)

With viscosity $\nu > 0$ we have the **Navier-Stokes equations**:

$$v_t + v \cdot \nabla v + \frac{1}{\rho} \nabla p = \nu \Delta v, \quad \nabla \cdot v = 0.$$  \(1.9\)

If we omit viscosity we have the **Euler equations**:

$$v_t + v \cdot \nabla v + \frac{1}{\rho} \nabla p = 0, \quad \nabla \cdot v = 0.$$  \(1.10\)

Each set of equations is complete in that it determines a unique solution subject to initial and boundary conditions, with qualifications. It appeared the pressure $p$ was an extra unknown. However, using the condition $\nabla \cdot v = 0$ we can apply $\nabla \cdot$ through the momentum equation and obtain

$$\Delta p = -\nabla \cdot (v \cdot \nabla v),$$  \(1.11\)
an equation which determines \( p \) from \( v \) subject to some boundary condition.

The scaling built into the Navier-Stokes equations is important practically and in verifying their validity. If we nondimensionalize with a typical length scale \( L \), velocity \( U \), and time \( L/U \), we find the nondimensionalized coefficient of viscosity is \( 1/R \), where

\[
R = LU/\nu
\]  

(1.12)
is a nondimensional quantity called the Reynolds number. This fact is used, for example, when the air flow around an airplane or SUV is studied by testing models in a wind tunnel; the model flow in the lab can be made to be a rescaling of the realistic flow on the large scale. The Reynolds number varies from \( R << 1 \) for a one-cell organism to \( R >> 10^6 \) for airplanes. This wide range corresponds to the great variety in behavior of fluids.

2 What we know about existence of solutions.

We can consider initial value problems in all space or in domains with boundaries (say with solid walls), with or without viscosity, in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). In 2D, for a reasonably good initial state, a good solution exists for all time. This is true for either the Euler or Navier-Stokes equations, though the techniques of proof are generally different. Perhaps this means that in a 2D world complicated creatures like us couldn’t exist!

In 3D, for the Euler equations, with or without boundaries, a good solution exists at least for a short time (depending on the size of the initial state). We don’t know whether or not singularities can form in finite time. “Most people” believe that singularities can form. There have been extensive numerical studies to search for such singularities. Neither analysis nor numerics are conclusive.

For the Navier-Stokes equations, a good solution exists at least for a short time. A “weak” solution exists for all time, but it is so weak that we don’t know that it is unique. For suitably small initial states, a good solution exists for all time. We don’t know whether an arbitrary good solution can form singularities in finite time. That is the question for which the Clay prize is offered. “Most people” believe that singularities don’t form, and good solutions remain good. There are many analytical results that say a weak solution is smooth except on a small set in space-time, but the small set could be dense.

Of course I am speaking loosely and leaving out qualifications for the sake of giving an overview. We will be more careful as we proceed. The beliefs I stated are not well-based mathematically and not unanimously held. In this short course we will prove some key results of existence and qualitative behavior of solutions. These arguments will illustrate the use of important ideas and theorems that you learn in other courses, especially functional analysis and properties of Sobolev spaces.

3 Facts about \( L^p \) spaces you didn’t learn in 241.

You are used to the Hölder inequality. We will use two straightforward generalizations. If \( p_1, p_2, p_3 \) are such that \((1/p_1) + (1/p_2) + (1/p_3) = 1\), then (it works for > 3 factors also)

\[
\int |f_1 f_2 f_3| \leq \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}.
\]  

(3.1)
If \((1/p) + (1/q) = (1/r)\), then
\[
\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.
\] (3.2)

We can interpolate between two \(p\)'s. If \(1 \leq p_1 \leq r \leq p_2 < \infty\) and \(\theta\) is defined by \(1/r = \theta/p_1 + (1 - \theta)/p_2\), then
\[
\|f\|_{L^r} \leq \|f\|_{L^{p_1}}^\theta \|f\|_{L^{p_2}}^{1-\theta}.
\] (3.3)

Two of these three facts are on p. 623 in Evans’ PDE.

4 The projection onto divergence-free vector fields.

It is natural to view the Euler or Navier-Stokes equations as evolution equations in a space of vector fields with divergence zero. Consider \(v \in L^2(\mathbb{R}^3; \mathbb{R}^3)\). We form the subspace \(H\) consisting of those \(v\) with divergence zero in the distributional sense. For \(v \in L^2\), \(\nabla \cdot v\) is the distribution, or generalized function, such that for every test function \(\phi \in C_0^\infty(\mathbb{R}^3)\),
\[
\langle \nabla \cdot v, \phi \rangle = -\langle v, \nabla \phi \rangle
\] (4.1)
where the right side is the integral of a scalar product and the left side is the action of the distribution on the test function. Note for smooth \(v\) this identity holds by the divergence theorem. Thus
\[
H = \{v \in L^2 : \langle v, \nabla \phi \rangle = 0 \ \forall \phi \in C_0^\infty(\mathbb{R}^3)\}\] (4.2)

The orthogonal complement of \(H\) in \(L^2\) is, obviously, the closure in \(L^2\) of the space of \(\nabla q\) with \(q \in C_0^\infty\). What is this closure? One case of the Sobolev inequalities (e.g. see Evans PDE, p. 263) says that
\[
\|q\|_{L^6} \leq \|\nabla q\|_{L^2}, \quad q \in C_0^\infty(\mathbb{R}^3)\] (4.3)

We may discuss the Sobolev inequalities more later. The number 6 is special to dimension 3. Let’s get our analytical muscles toned up for the challenges ahead by proving

Lemma 4.1 The orthogonal complement of \(H\) as a subspace of \(L^2(\mathbb{R}^3; \mathbb{R}^3)\) is
\[
G = \{\nabla q : q \in L^6 \text{ and } \nabla q \in L^2\}.
\] (4.4)

Proof. We have to show that the closure mentioned is the space \(G\) defined above. The closure is contained in \(G\), by (4.3) and the completeness of the \(L^p\) spaces. To prove \(G\) is contained in the closure, suppose \(q \in G\). We use the fact that \(C_0^\infty\) is dense in the \(L^p\) spaces. We first approximate \(q\) by a function of compact support. Choose a smooth function \(\zeta : \mathbb{R}^3 \to [0, 1]\) so that \(\zeta(x) = 1\) for \(|x| \leq 1\) and \(\zeta(x) = 0\) for \(x \geq 2\). (It could be a function of \(|x|\) alone.) Define \(\zeta_n(x) = \zeta(x/n)\) and \(q_n(x) = \zeta_n(x)q(x)\). Clearly \(q_n \to q\) in \(L^6\). Now \(\nabla q_n = \zeta_n \nabla q + (\nabla \zeta_n)q\). The first term goes to \(\nabla q\) in \(L^2\) as \(n \to \infty\). For the other term, note that \(|\nabla \zeta_n(x)| \leq C_1/n\), and \(\nabla \zeta_n(x) \neq 0\) only for \(n \leq |x| \leq 2n\), so that
\[
\int |\nabla \zeta_n|^3 \leq C (1/n)^3 \cdot n^3 \leq C_2
\]
and therefore, using (3.2),
\[ \|(\nabla \zeta_n)q\|_{L^2} \leq C\|\nabla \zeta_n\|_{L^3}\|q\|_{L^6(|x|>n)} \leq C'\|q\|_{L^6(|x|>n)} \to 0 \text{ as } n \to \infty. \]
Thus \(q_n\) has compact support and, for \(n\) large, \(q_n\) is close to \(q\) in \(L^6\), and \(\nabla q_n\) is close to \(\nabla q\) in \(L^2\). For each \(n\) we can find a \(C_0^\infty\) function close in both senses. We can do both at the same time by mollification (convolution with a smooth function), which we will discuss soon. □

We will use the orthogonal projection \(P\) of \(L^2\) on \(H\); \(I-P\) is the projection on \(G\). What are they? For \(w \in L^2\) we have a unique decomposition
\[ w = v + \nabla q = Pw + (I-P)w. \tag{4.5} \]
Applying \(\nabla \cdot\) through the equation and using \(\nabla \cdot v = 0\), we get
\[ \Delta q = \nabla \cdot w; \tag{4.6} \]
this should determine \(q\). To be more careful, \(\nabla q = (I-P)w\) is defined by the fact that \(v = w - \nabla q\) is orthogonal to \(G\), or
\[ \langle \nabla q, \nabla \phi \rangle = \langle w, \nabla \phi \rangle \quad \forall \nabla \phi \in G \tag{4.7} \]
and in particular for \(\phi \in C_0^\infty\). Thus
\[ -\langle q, \Delta \phi \rangle = \langle w, \nabla \phi \rangle \tag{4.8} \]
for test functions \(\phi\), showing that (4.6) holds in the distributional sense.

It is helpful to write the definition of \(\nabla q\) in the Fourier transform. From (4.6) we have
\[ -|k|^2 \hat{q}(k) = ik \cdot \hat{w}(k) \] and therefore
\[ (\nabla q)^\wedge(k) = k\frac{k \cdot \hat{w}(k)}{|k|^2}. \tag{4.9} \]
From this it is apparent that \(\|\nabla q\|_{L^2} \leq C\|w\|_{L^2}\), as we expect. In the transform \((I-P)\) projects \(\hat{w}(k)\) along \(k\). Of course we can write a similar formula for \((Pw)^\wedge(k)\).

For later use we note that \(P\) commutes with derivatives, i.e., if \(D\) is any partial derivative, then \(PDw = DPw\). This is apparent since \(Dw = Dv + \nabla(Dq)\) and \(\nabla \cdot Dv = D\nabla \cdot v = 0\). This is NOT true in domains with boundaries, since \(P\) includes a boundary condition, as we shall see.

5 The basic energy estimate.

The quantity
\[ \frac{1}{2} \int_{\mathbb{R}^3} |v|^2, \tag{5.1} \]
represents the kinetic energy in the fluid, depending on time \(t\). (We will assume here the density is constant and set it to 1.) We estimate the rate of change of this energy from the
equations (1.9) or (1.10). At first we assume all functions are smooth enough and approach zero at infinity fast enough for our purposes. Then we will discuss the justification of the steps. We often use the divergence theorem and two consequences called Green’s first and second identities: If \( u, v \), are any \( C^1 \) vector fields on a domain \( D \) with reasonable boundary, then

\[
\int_D u \Delta v + \int_D \nabla u \cdot \nabla v = \int_{\partial D} u \frac{\partial v}{\partial n} \, dS,
\]

\( (5.2) \)

\[
\int_D u \Delta v - \int_D v \Delta u = \int_{\partial D} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) \, dS.
\]

\( (5.3) \)

I’ll refer to these as G1 and G2 for short. To use these in all \( \mathbb{R}^3 \) we might choose \( D \) to be a ball of radius \( R \), apply the identity for this \( D \), and then take the limit as \( R \to \infty \) if possible.

Now we begin the energy estimate. We will write \( v^2 \) for \( |v|^2 = v \cdot v \). Since \( (d/dt)(|v|^2/2) = v_t \cdot v \), we take the scalar product of (1.9) with \( v \) and integrate to get

\[
\frac{1}{2} \frac{d}{dt} \int v^2 + \int (v \cdot \nabla v) \cdot v + \int \nabla p \cdot v = \nu \int \Delta v \cdot v.
\]

\( (5.4) \)

The last term is a sum (for the dot product) of terms to which G1 applies, so that

\[
\nu \int \Delta v \cdot v = -\nu \int |\nabla v|^2
\]

\( (5.5) \)

with a double sum on the right. The integrand \( \nabla p \cdot v \) is one term in \( \nabla \cdot (pv) \), and by the divergence theorem

\[
\int \nabla p \cdot v = -\int p \nabla \cdot v = 0.
\]

\( (5.6) \)

The nonlinear term is more interesting. Let’s write out the sums, using \( D_j = \partial / \partial x_j \) and \( v_{i,j} = \partial v_i / \partial x_j \),

\[
\int (v \cdot \nabla v) \cdot v = \sum_{i,j} v_j v_{i,j} v_i = \sum_{i,j} (D_j v_i) (v_j v_i)
\]

\( (5.7) \)

\[
= -\sum_{i,j} v_i (v_{j,j} v_i + v_j v_{i,j}) = 0 - \int v \cdot (v \cdot \nabla v)
\]

\( (5.8) \)

using integration by parts (or the divergence theorem) and the fact that \( \nabla \cdot v = 0 \). Thus the integral is minus itself, and

\[
\int (v \cdot \nabla v) \cdot v = 0.
\]

\( (5.9) \)

We have reached the important conclusion

\[
\frac{1}{2} \frac{d}{dt} \int v^2 = -\nu \int |\nabla v|^2.
\]

\( (5.10) \)

Without boundaries, energy can dissipate through viscosity, and otherwise it can’t change.

When is this justified? It will be convenient to use the Sobolev space \( H^m(\mathbb{R}^3) \) consisting of functions in \( L^2 \) whose (distributional) derivatives of order \( \leq m \) are in \( L^2 \); this is meaningful
for $m \geq 0$ an integer, and there are natural extensions to other $m$ using the Fourier transform. We know that $C^\infty_0$ is dense in $H^m$. We will not distinguish explicitly between vector and scalar functions. At first we ignore dependence on $t$.

We will start with statement that includes the treatment of the nonlinear term above: Suppose $f, g$ are scalar functions and $u$ is a vector field. Then

$$\nabla \cdot (fgu) = (u \cdot \nabla f)g + (u \cdot \nabla g)f + fg(\nabla \cdot u). \tag{5.11}$$

If $f, g, u$ are all $C^\infty_0$ we can use the divergence theorem to say

$$0 = \int (u \cdot \nabla f)g + \int (u \cdot \nabla g)f + \int fg(\nabla \cdot u). \tag{5.12}$$

Now suppose $f, g, u$ are in $H^1$. We can find sequences $f_n, g_n, u_n$ in $C^\infty_0$ so that $f_n \to f$ in $H^1$ and the same for $g, u$. It follows from the Sobolev inequality (4.3) that $f_n \to f$ in $L^6$. (Check this for yourself using Cauchy sequences.) Since $f_n \to f$ in $L^2$ as well, we can say $f_n \to f$ in $L^4$ using (3.3). In the same way, $u_n \to u$ in $L^4$. Now, with these and $\nabla g_n \to \nabla g$ in $L^2$, we can use (3.1) to show

$$\int (u_n \cdot \nabla g_n)f_n \to \int (u \cdot \nabla g)f. \tag{5.13}$$

(You check this.) Similarly we can write the other two integrals in (5.12) as limits and thereby show that (5.12) holds for $f, g, u$ in $H^1$. In particular if $u = v$ and $\nabla \cdot v = 0$, we get

$$\int (v \cdot \nabla f)g = -\int f(v \cdot \nabla g), \quad \int (v \cdot \nabla f) = 0. \tag{5.14}$$

This justifies (5.9) for $v \in H^1$ with $\nabla \cdot v = 0$; we can let $f$ above be one of the components $v_i$ of $v$ and then sum.

In a similar but easier way, (5.5) is justified for $v \in H^2$. Similar remarks apply to the two Green’s identities. Again by a limiting argument, (5.6) is justified for $\nabla p$ in the space $G$ we defined before and $v \in L^2$ with $\nabla \cdot v = 0$.

We will require $v$ to be in a space such as $C(0, T; H^2)$, meaning $v(t)$ depends continuously on $t$ for $t \in [0, T]$, with values in $H^2(\mathbb{R}^3)$. We can see now that for the Navier-Stokes equations (1.9), the energy estimate is justified if $v \in C^1(0, T; L^2) \cap C(0, T; H^2)$. For the Euler equations (1.10), with $\nu = 0$, we can replace $H^2$ by $H^1$. This is the first step toward proving that solutions exist.

### 6 The existence theorem in $\mathbb{R}^3$.

We will prove the following existence theorem for the Euler and Navier-Stokes equations.

**Theorem 6.1** Given $v_0 \in H^m(\mathbb{R}^3; \mathbb{R}^3)$ with $\nabla \cdot v_0 = 0$, $m \geq 3$, there is a $T > 0$ so that the initial value problem

$$v_t + v \cdot \nabla v + \nabla p = \nu \Delta v, \quad \nabla \cdot v = 0, \quad v(0) = v_0$$

has a unique solution with $v \in C(0, T; H^m(\mathbb{R}^3; \mathbb{R}^3)) \cap C^1(0, T; H^{m-2}(\mathbb{R}^3; \mathbb{R}^3))$ and $\nabla p \in C(0, T; G \cap H^{m-2}(\mathbb{R}^3; \mathbb{R}^3))$. For the Euler equations (with $\nu = 0$), $m - 2$ can be replaced with $m - 1$. 

We need \( m \geq 3 \) because estimates for nonlinear terms are easier with higher norms. The time \( T \) will have the form \( C/\|v_0\|_{H^3} \); it is limited by growth due to nonlinearities. \( G \) is the space of gradients introduced before. With \( \nu > 0 \) the solution is actually smooth for \( t > 0 \).

Here is an outline of the proof:

1. We first find estimates for the growth in time of a solution in the \( H^m \)-norm, presuming that one exists; this is called an a priori estimate.
2. We approximate the equations with a smoothing of order \( \varepsilon \) and show these equations have a solution.
3. We find estimates for the solutions of the \( \varepsilon \)-equations independent of \( \varepsilon \).
4. We show that a limit exists as \( \varepsilon \to 0 \) and the limit solves the original problem.

Actually (1) is not logically necessary, but it is a step toward doing (3) and is a good way to begin such a proof.

### 7 Facts about Sobolev spaces.

We use the multi-index notation for derivatives. For a function \( f \) on \( \mathbb{R}^n \) write \( D^\alpha f \), with \( \alpha = (\alpha_1, \ldots, \alpha_n) \), meaning the partial derivative \( \alpha_1 \) times in \( x_1 \) etc. The total order of the derivative is \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), which we write as \( |\alpha| \).

Thus for instance the Sobolev space \( H^m(\mathbb{R}^n) \), \( m \geq 0 \) an integer is

\[
H^m(\mathbb{R}^n) = \{ u : D^\alpha u \in L^2(\mathbb{R}^n) \ \forall \alpha, |\alpha| \leq m \}.
\]

We will sometimes use the \( H^m \) inner product

\[
\langle f, g \rangle = \sum_{|\alpha| \leq m} \langle D^\alpha f, D^\alpha g \rangle
\]

We will use some important facts about Sobolev spaces:

1. (Sobolev’s Theorem) For \( m > n/2 \), if \( f \in H^m(\mathbb{R}^n) \), then \( f \) is continuous and bounded, and

\[
\|f\|_{L^\infty} \leq C \|f\|_{H^m}.
\]

2. For \( m > n/2 \), if \( f, g \in H^m(\mathbb{R}^n) \), then the product \( fg \) is in \( H^m(\mathbb{R}^n) \), and

\[
\|fg\|_{H^m} \leq C \|f\|_{H^m} \|g\|_{H^m}.
\]

3. For integer \( m \geq 0 \), if \( f, g \in H^m(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), then \( fg \in H^m(\mathbb{R}^n) \), and

\[
\|fg\|_{H^m} \leq C \left( \|f\|_{H^m} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{H^m} \right).
\]

4. For integer \( m \geq 0 \), if \( f, g \in H^m(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), if \( |\alpha| \leq m \),

\[
\|D^\alpha (fg) - fD^\alpha g\|_{L^2} \leq C \left( \|f\|_{H^m} \|g\|_{L^\infty} + \|\nabla f\|_{L^\infty} \|g\|_{H^{m-1}} \right).
\]

Fact (1) is proved in Folland’s book simply using the Fourier transform. Fact (2) can be proved in a similar way; I’ll give you a note on request. Adams’ book on Sobolev spaces does much like this in detail. Evans’ book, Sec. 5.6, is a good reference for the Sobolev inequalities and a more general version of (1). The proof of (3) and (4) is based on the Gagliardo-Nirenberg inequalities; see Taylor, Vol. 3, p. 10, Prop. 3.7.
8 The a priori estimate.

Suppose \( v \in C(0, T; H^{m+1}(\mathbb{R}^3; \mathbb{R}^3)) \cap C^1(0, T; H^m(\mathbb{R}^3; \mathbb{R}^3)) \) is a solution of the Euler equations in the form

\[
v_t + P(v \cdot \nabla v) = 0, \quad \nabla \cdot v(t) = 0
g
\]

with the initial condition

\[
v(0) = v_0 \in H^{m+1}, \quad \nabla \cdot v(0) = 0.
\]

We will estimate the growth in time of \( \|v\|_{H^m} = \|v\|_m \). We can’t estimate \( \|v\|_{m+1} \) directly! For the moment, we are giving ourselves an “extra” derivative to use. We proceed as in the basic energy estimate, putting in derivatives.

We apply \( D^\alpha \) through (8.1), \( |\alpha| \leq m \), take the dot product with \( D^\alpha v \), and then integrate. We write the integral as an \( L^2 \)-inner product. We get

\[
\langle D^\alpha v_t, D^\alpha v \rangle + \langle D^\alpha P(v \cdot \nabla v), D^\alpha v \rangle = 0.
\]

It will be important here that \( D^\alpha \) and \( P \) commute since there is no boundary. Thus \( PD^\alpha v = D^\alpha Pv = D^\alpha v \). Also \( P \) is self-adjoint. Thus, since \( Pv = v \),

\[
\langle D^\alpha P(v \cdot \nabla v), D^\alpha v \rangle = \langle PD^\alpha (v \cdot \nabla v), D^\alpha v \rangle = \langle D^\alpha (v \cdot \nabla v), PD^\alpha v \rangle = \langle D^\alpha (v \cdot \nabla v), D^\alpha v \rangle.
\]

The last term leads to several terms when we apply \( D^\alpha \) to the product. The term with the highest order derivative, possibly \( m+1 \), occurs when \( D^\alpha \) hits \( \nabla v \). We note that by (5.14)

\[
\langle v \cdot \nabla D^\alpha v, D^\alpha v \rangle = 0
\]

and rewrite (8.3), using (8.4) and subtracting (8.5),

\[
\frac{1}{2} \frac{d}{dt} \|D^\alpha v\|_0^2 + \langle D^\alpha (v \cdot \nabla v) - v \cdot \nabla D^\alpha v, D^\alpha v \rangle = 0.
\]

Now we can use (7.6) with \( f \) replaced by \( v, g \) by \( \nabla v \), to estimate

\[
\|D^\alpha (v \cdot \nabla v) - v \cdot \nabla D^\alpha v\|_0 \leq C (\|v\|_m \|\nabla v\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|\nabla v\|_{m-1}) = C' \|\nabla v\|_{L^\infty} \|v\|_m
\]

so that

\[
\frac{1}{2} \frac{d}{dt} \|D^\alpha v\|_0^2 \leq C' \|\nabla v\|_{L^\infty} \|v\|_m \|D^\alpha v\|_0 \leq C' \|\nabla v\|_{L^\infty} \|v\|_m^2
\]

and summing over \( \alpha, |\alpha| \leq m \),

\[
\frac{d}{dt} \|v\|_m^2 \leq C \|\nabla v\|_{L^\infty} \|v\|_m^2.
\]

Now if \( m > (3/2) + 1 \) and \( y(t) = \|v(t)\|_m^2 \), we can use (7.3) to say

\[
\frac{dy}{dt} \leq Cy^{3/2}, \quad y(0) = \|v_0\|_m^2.
\]

Solving this inequality, we get a time-dependent upper bound for \( y(t) \) which goes to infinity at a time proportional to \( y(0)^{-1/2} \) or \( \|v_0\|_m^{-1} \). Choosing a time \( T_0 \) before that, we can conclude that there exist \( M \) and \( T_0 \), depending on \( \|v_0\|_m \), so that \( \|v(t)\|_m \leq M \) for \( 0 \leq t \leq T_0 \). With slightly more care we can get \( T_0 \) independent of \( m \) but depending on \( \|v_0\|_{m_0} \) for one \( m_0 > (3/2) + 1 \).
9 Mollification or smoothing.

We write the Fourier transform for \( f : \mathbb{R}^n \rightarrow \mathbb{C}^n \) as
\[
\hat{f}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ikx} \, dx, \quad f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(k)e^{-ikx} \, dk.
\] (9.1)

For mollifying or smoothing we choose a function \( \zeta \in \mathcal{S} \), the Schwartz class of smooth, rapidly decreasing functions, with \( \int_{\mathbb{R}^n} \zeta = 1 \). For simplicity we will assume \( \zeta(x) \geq 0 \). For \( \varepsilon > 0 \) we define \( \zeta_{\varepsilon}(x) = \varepsilon^{-n}\zeta(x/\varepsilon) \). Note \( \int_{\mathbb{R}^n} \zeta_{\varepsilon} = 1 \). (We could choose \( \zeta(x) = ce^{-x^2} \), for example, where \( c \) is the normalizing constant. Later it will be convenient to assume \( \zeta \) is chosen so that \( \zeta(x) = 0 \) for \( |x| \geq 1 \).)

Given \( f \in L^p(\mathbb{R}^n) \), for any \( p \), we define a smooth approximation \( S_{\varepsilon}f \) as
\[
(S_{\varepsilon}f)(x) = \int_{\mathbb{R}^n} \zeta_{\varepsilon}(x-y)f(y) \, dy.
\] (9.2)

This is the convolution \( \zeta_{\varepsilon} \ast f \). Since derivatives commute with convolution,
\[
D^\alpha S_{\varepsilon}f = S_{\varepsilon}(D^\alpha f)
\] (9.3)
provided \( D^\alpha f \in L^p \) and even more generally. Several properties will be useful:
\[
\|S_{\varepsilon}f\|_{L^p} \leq \|f\|_{L^p}, \quad \|S_{\varepsilon}f\|_{H^m} \leq \|f\|_{H^m}.
\] (9.4)

The first holds because \( \|\zeta_{\varepsilon}\|_{L^1} = 1 \) and the convolution of an \( L^1 \) with an \( L^p \) is bounded in \( L^p \) by the product of the norms. The second follows from (9.3).

(1) For \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \),
\[
S_{\varepsilon}f \rightarrow f \quad \text{in} \ L^p \quad \text{as} \ \varepsilon \rightarrow 0.
\] (9.5)
(See Folland p. 11; Evans p. 629; or Majda and Bertozzi p. 97, 131.) We prove this for \( p = 2 \) using the Fourier transform. Note \( \check{\zeta}_{\varepsilon}(k) = \zeta(\varepsilon k) \), and \( \hat{\zeta}(0) = (2\pi)^{-n/2} \) pointwise in \( k \) as \( \varepsilon \rightarrow 0 \). For \( f \in L^2 \),
\[
(S_{\varepsilon}f)(k) = (\zeta_{\varepsilon} \ast f)(k) = (2\pi)^{n/2}\hat{\zeta}_{\varepsilon}(k)\hat{f}(k) = (2\pi)^{n/2}\hat{\zeta}(\varepsilon k)\hat{f}(k)
\] (9.6)
and
\[
(S_{\varepsilon}f)(k) - \hat{f}(k) = (2\pi)^{n/2}(\hat{\zeta}(\varepsilon k) - \hat{\zeta}(0))\hat{f}(k).
\] (9.7)
Since the Fourier transform preserves the \( L^2 \) norm, \( \|S_{\varepsilon}f - f\|_{L^2}^2 \) is the integral of the absolute square of the above, and this integral goes to zero by the Lebesgue Dominated Convergence Theorem, establishing (9.5) for \( p = 2 \).

(2) For \( f \in H^m(\mathbb{R}^n) \), \( m \geq 0 \) integer,
\[
S_{\varepsilon}f \rightarrow f \quad \text{in} \ H^m.
\] (9.8)
This follows by combining (9.5) with (9.3), using the definition of \( H^m \).
(3) If \( f \in H^1 \), then
\[
\|S_\varepsilon f - f\|_{L^2} \leq C\varepsilon\|f\|_{H^1} .
\] (9.9)
From (9.7), \( |(S_\varepsilon f)'(k) - \hat{f}(k)| \leq C\varepsilon|k| \left( \sup |\nabla \hat{\zeta}| \right) |\hat{f}(k)| = C'\varepsilon|k| |\hat{f}(k)| \). Now square and integrate to get (9.9).

(4) If \( f \in H^m \), then
\[
\|S_\varepsilon f - f\|_{H^{m-1}} \leq C\varepsilon\|f\|_{H^m} .
\] (9.10)
This follows from (9.9), using (9.3) once again.

(5) If \( f \in H^m \) and \( \ell \geq 0 \), then
\[
\|S_\varepsilon f\|_{H^{m+\ell}} \leq C\varepsilon^{-\ell}\|S_\varepsilon f\|_{H^m} .
\] (9.11)
That is we gain a derivative at a cost of \( \varepsilon^{-1} \). To see this we write
\[
\|S_\varepsilon f\|_{H^{m+\ell}}^2 = C \int |\hat{\zeta}(\varepsilon k)|^2|\hat{f}(k)|^2(1 + |k|^2)^{m+\ell} dk
\]
\[
= C\varepsilon^{-2\ell} \int |\hat{\zeta}(\varepsilon k)|^2\varepsilon^{2\ell}(1 + |k|^2)^{\ell}|\hat{f}(k)|^2(1 + |k|^2)^m dk
\]
\[
\leq C\varepsilon^{-2\ell} \int |\hat{\zeta}(\varepsilon k)|^2(1 + |\varepsilon k|^2)^{\ell}|\hat{f}(k)|^2(1 + |k|^2)^m dk
\]
\[
\leq C'\varepsilon^{-2\ell} \int |\hat{f}(k)|^2(1 + |k|^2)^m dk = C'\varepsilon^{-2\ell}\|f\|_{H^m}^2 .
\] (9.12)
In the last step we use the fact that \( \hat{\zeta} \) is rapidly decreasing.

10 The regularized equations.

We will use this \( \varepsilon \)-regularization of the Navier-Stokes equations with velocity \( v \) replaced by an unknown \( v^\varepsilon \),
\[
v^\varepsilon_t + S_\varepsilon P \left[ (S_\varepsilon v^\varepsilon) \cdot \nabla (S_\varepsilon v^\varepsilon) \right] = \nu S_\varepsilon^2 \Delta v^\varepsilon .
\] (10.1)
With \( v_0 \) specified in \( H^m \) at least, we impose the initial condition
\[
v^\varepsilon(0) = S_\varepsilon v_0 .
\] (10.2)
The \( S_\varepsilon \) factors are placed in the equation so that we can estimate as we did for the exact equations in Sec. 8. We will show in the next section that a solution exists. Assuming for now that solutions exist, we apply \( D^\alpha \), take the scalar product with \( D^\alpha v^\varepsilon \), integrate, and sum over \( \alpha , |\alpha| \leq m \), noticing what happens to the \( S_\varepsilon \)'s in the nonlinear and viscosity terms. It is important that \( S_\varepsilon \) is self-adjoint and commutes with derivatives. We obtain
\[
\frac{1}{2} \frac{d}{dt}\|v^\varepsilon\|_{H^m}^2 + \nu \|\nabla S_\varepsilon v^\varepsilon\|_{H^m}^2 \leq C\|\nabla (S_\varepsilon v^\varepsilon)\|_{L^\infty} \|v^\varepsilon\|_{H^m}^2 .
\] (10.3)
We have used (9.4) in the last factor. For the special case with \( \alpha = 0 \), we can estimate as in the original energy inequality. The nonlinear term does not contribute, and we see the energy is nonincreasing:
\[
\|v^\varepsilon(t)\|_{L^2} \leq \|v^\varepsilon(0)\|_{L^2} \leq C\|v_0\|_{L^2} .
\] (10.4)
The constant in (10.3) is independent of $\varepsilon$; we now improve the estimate, but in a way that depends on $\varepsilon$. We use Sobolev’s Theorem (7.3) and the smoothing property (9.11) of $S_\varepsilon$:

$$
\|\nabla (S_\varepsilon v^\varepsilon)\|_{L^\infty} \leq C_1 \|S_\varepsilon v^\varepsilon\|_{H^3} \leq C_2 \varepsilon^{-3} \|v^\varepsilon\|_{L^2} \leq C_3 \varepsilon^{-3} \|v_0\|_{L^2}
$$

(10.5)

using (10.4). Thinking of $\|v_0\|_{L^2}$ as a constant, we simplify (10.3) to

$$
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{H^m} \leq C_0 \varepsilon^{-3} \|v^\varepsilon\|^2_{H^m}
$$

(10.6)

from which we get

$$
\|v^\varepsilon(t)\|_{H^m} \leq e^{C_0 t/\varepsilon^3} \|v^\varepsilon(0)\|_{H^m}.
$$

(10.7)

Using $\|v^\varepsilon(0)\|_{H^m} \leq C \|v_0\|_{H^m}$, and assuming the solution exists up to some time $T$, we can conclude that for $0 \leq t \leq T$,

$$
\|v^\varepsilon(t)\|_{H^m} \leq C e^{C_0 T/\varepsilon^3} \|v_0\|_{H^m} \equiv R(\varepsilon, T).
$$

(10.8)

The estimate (10.8) is good for all time, but has awful dependence on $\varepsilon$. In contrast, we can also use (10.3) to get an estimate independent of $\varepsilon$ for a short time. In place of (10.5) we have

$$
\|\nabla (S_\varepsilon v^\varepsilon)\|_{L^\infty} \leq \|S_\varepsilon (\nabla v^\varepsilon)\|_{L^\infty} \leq \|\nabla v^\varepsilon\|_{L^\infty}
$$

(10.9)

so that

$$
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{H^m} \leq C \|\nabla v^\varepsilon\|_{L^\infty} \|v^\varepsilon\|^2_{H^m} \leq C_2 \|v^\varepsilon\|_{H^3} \|v^\varepsilon\|^2_{H^m}
$$

(10.10)

and with $y(t) = \|v^\varepsilon(t)\|^2_{H^m}$ as before, $dy/dt \leq C_3 y^{3/2}$. From this we see that $y(t)$ is bounded for a short time $T_0$ (independent of $\varepsilon$) depending on $\|v_0\|_{H^m}$. We have reached this important conclusion:

There exist $T_0$ and $R_0$, depending only on $\|v_0\|_{H^m}$, so that, if we have a solution of (10.1) and (10.2) for some $\varepsilon > 0$ and for $0 \leq t \leq T_*$, for some $T_* \leq T_0$, then

$$
\|v^\varepsilon(\cdot, t)\|_{H^m} \leq R_0, \quad 0 \leq t \leq T_*.
$$

(10.11)

11 Existence for the regularized equations.

We will now prove that solutions exist for the regularized equations. With $\varepsilon > 0$ fixed, our equation (10.1) is “like” an ordinary differential equation, because operators such as $D^\alpha S_\varepsilon$ are bounded, unlike the exact differential operator. We think of the equation as

$$
v^\varepsilon_t = F(v^\varepsilon), \quad F(u) = -S_\varepsilon P(S_\varepsilon u) \cdot \nabla (S_\varepsilon u) + \nu S_\varepsilon^2 \Delta u.
$$

(11.1)

We will use bounds on $F$ and its difference quotients for $u$ in a set bounded in $m$-norm by some radius $\rho$. From now on we choose $T$ in (10.8) to be $T_0$ as in (10.11), independent of $\varepsilon$. We can estimate

$$
\|\Delta S_\varepsilon u\|_{H^m} \leq C \varepsilon^{-2} \|u\|_{H^m}
$$

(11.2)
Similarly, in this way we can show that, for \( u, v \in H^m \) with \( \|u\|_H^m \leq \rho \) and \( \|v\|_H^m \leq \rho \),
\[
\|F(u) - F(v)\|_{H^m} \leq C_1 \varepsilon^{-1} (\|u\|_{H^m} + \|v\|_{H^m}) \|u - v\|_{H^m} + C_2 \nu \varepsilon^{-2} \|u - v\|_{H^m}
\]  
(11.3)
or
\[
\|F(u) - F(v)\|_{H^m} \leq (2C_1 \rho \varepsilon^{-1} + C_2 \nu \varepsilon^{-2}) \|u - v\|_{H^m}
\]  
(11.4)
and similarly
\[
\|F(u)\|_{H^m} \leq C'_1 \varepsilon^{-1} \rho^2 + C'_2 \varepsilon^{-2} \rho.
\]  
(11.5)
We use these to prove the following existence theorem:

**Theorem 11.1** With \( T_0 \) and \( R_0 \) chosen as in (10.11), for each \( \varepsilon > 0 \) small there is a \( v^\varepsilon \in C^1(0,T_0;H^m) \) solving the \( \varepsilon \)-problem (10.1), (10.2) and obeying the estimate (10.11).

To prove this we use an existence theorem which is the infinite dimensional version of the usual existence theorem for ordinary differential equations. It can be proved by the same existence argument based on Picard iteration that is given in many textbooks.

**Theorem 11.2** Let \( X \) be a Banach space. Suppose there are \( R, K, M > 0 \) and a function \( F : \{u \in X : \|u\|_X < 2R\} \to X \) so that
(i) \( \|F(u)\|_X \leq M \) for \( \|u\|_X < 2R \);
(ii) \( \|F(u) - F(v)\|_X \leq K \|u - v\|_X \) for \( \|u\|_X < 2R, \|v\|_X < 2R \).

Then for any \( u_0 \in X \) with \( \|u_0\|_X < R \), the initial value problem
\[
\frac{du}{dt} = F(u), \quad u(0) = u_0
\]  
(11.6)
has a unique solution \( u \in C^1(0,T;X) \) with \( T = R/M \).

**Proof of Theorem 11.1.** We determine \( T_0 \) and \( R_0 \) from (10.11). To apply Theorem 11.2 to the \( \varepsilon \)-problem, first choose \( R = R_0 + 1 \) in the hypothesis and \( X = H^m \). Then determine \( M \) and \( K \), according to (11.4) and (11.5), with \( \rho = 2R \), so that (i) and (ii) hold in Theorem 11.2. \( M \) and \( K \) depend on \( \varepsilon \). We conclude from Thm. 11.2 that there is a solution of (10.1), (10.2) for time \( T(\varepsilon) = R/M \). For \( \varepsilon \) small, \( T(\varepsilon) \) can be very small. However, from (10.11) we know that if \( T(\varepsilon) \leq T_0 \), then \( \|v^\varepsilon(t)\|_m \leq R_0 \) for \( t \leq T(\varepsilon) \). We can now use the ODE existence theorem *again*, starting at time \( T(\varepsilon) \), with the *same* choice of \( R, K, M \), and thus extending the solution by a further time \( T(\varepsilon) = R/M \) to time \( 2T(\varepsilon) \). We can continue with further steps of length \( T(\varepsilon) \) until we reach the time \( T_0 \) and obtain the solution in \( v^\varepsilon \in C^1(0,T_0;H^m) \), which satisfies (10.11) by the choice of \( T_0 \).

### 12 Facts about weak convergence.

Suppose \( H \) is a Hilbert space. We say a sequence \( f_n \) **converges strongly** to a limit \( f \) in \( H \) if it converges in norm, that is, \( \|f_n - f\|_H \to 0 \) as \( n \to \infty \). We say a sequence \( f_n \) **converges weakly** to \( f \) if, for every \( g \in H \), \( \langle f_n, g \rangle \to \langle f, g \rangle \) as \( n \to \infty \). For example, the sequence \( \sin nx \) converges to 0 weakly in \( L^2(0,\pi) \) but not strongly. Sometimes we write \( f_n \to f \) for strong convergence and \( f_n \rightharpoonup f \) for weak convergence, but the difference between the two...
arrows is easy to miss. Sometimes we say “\( f_n \to f \) weakly”. If I use an ordinary arrow, I mean strong unless I say weak. Some important facts:

1. If \( f_n \to f \), then \( \|f\| \leq \lim \inf \|f_n\| \).

2. If \( f_n \to f \), then \( f_n \to f \) if and only if \( \|f_n\| \to \|f\| \).

3. If a sequence \( f_n \) is bounded in norm, i.e. there is \( M > 0 \) so that \( \|f_n\| \leq M \) for each \( n \), and if \( \langle f_n, g \rangle \to \langle f, g \rangle \) for each \( g \) in a dense subset of \( H \), then \( f_n \to f \).

4. If a sequence \( f_n \) converges weakly, then it is bounded in norm. (This follows from the Uniform Boundedness Principle.)

5. If \( f_n \to f \) and \( g_n \to g \), then \( \langle f_n, g_n \rangle \to \langle f, g \rangle \). (This is not hard given (4).)

6. If the sequence \( f_n \) is bounded in norm, then it has a subsequence that converges weakly. Fact (6) is very useful to us. The conclusion is similar to that of the Banach-Alaoglu Theorem, but the proof can be much more direct. I posted a proof assuming \( H \) is separable.

### 13 Convergence of \( v^\varepsilon \).

We found that with \( T_0 \) as in (10.11), for \( 0 < \varepsilon \leq 1 \) the regularized problem (10.1), (10.2) has a solution bounded independent of \( \varepsilon \) in the space \( C(0, T; H^m) \) and therefore bounded in the Hilbert space \( L^2(0, T; H^m) \). Then by Fact (6) above there is a sequence \( \varepsilon_n \to 0 \) so that \( v^n \) converges weakly to a limit \( v \) in \( L^2(0, T; H^m) \), where \( v^n \) means \( v^\varepsilon_n \). We will show that this \( v \) solves the original problem and is in \( C(0, T; H^m) \).

From (10.11) we know that

\[
\|v^n(\cdot, t)\|_m \leq R_0, \quad 0 \leq t \leq T_0.
\]

We first show that

\[
\|v(\cdot, t)\|_m \leq R_0, \quad \text{for almost all } t.
\]

To prove this, it is enough to show that \( \forall \delta > 0 \) the set \( J_\delta = \{ t : \|v(\cdot, t)\|_m \geq R_0 + \delta \} \) has measure zero. Since \( v^n \to v \), it is also true that \( v^n \chi_{J_\delta} \to v \chi_{J_\delta} \). We apply Fact (1) to this sequence:

\[
(R_0 + \delta)^2 \mu(J_\delta) \leq \int_0^{T_0} \|v(\cdot, t)\chi_{J_\delta}(t)\|^2_m = \|v \chi_{J_\delta}\|^2_{L^2(H^m)} \leq \lim \inf \|v^n \chi_{J_\delta}\|^2_{L^2(H^m)} \leq R_0^2 \mu(J_\delta),
\]

a contradiction unless \( \mu(J_\delta) = 0 \). Thus (13.2) holds and \( v \in L^\infty(0, T; H^m) \).

As we will see, for a linear PDE weak convergence would be good enough to verify that the limit solves the original problem, but since our problem is nonlinear we need strong convergence in some norm. We use two kinds of compactness in two important theorems:

**Theorem 13.1 (Rellich Compactness Theorem)** If \( m > 0 \), \( 0 < \ell \leq m \), and \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), then every bounded sequence in \( H^m(\Omega) \) has a subsequence converging strongly in \( H^{m-\ell}(\Omega) \).

We combine this with a general version of the Arzela-Ascoli Theorem (e.g. see Royden, p. 169) to formulate this criterion for strong convergence:
Theorem 13.2 Suppose \( f_n \) is a sequence in \( C(0, T; H^m(\mathbb{R}^n)) \), \( m > 0 \). Suppose there are constants \( K, M > 0 \) and integer \( 0 < \ell \leq m \) so that \( \|f_n(t)\|_m \leq M \) for each \( n \) and \( 0 \leq t \leq T \), and \( \|f_n(s) - f_n(t)\|_{m-\ell} \leq K|s - t| \) for each \( n \) and \( 0 \leq s, t \leq T \). Then there is a subsequence \( f_{n_k} \) and \( f \in C(0, T; H^{m-\ell}_{loc}(\mathbb{R}^n)) \) so that for every \( \rho > 0 \), \( f_{n_k} \to f \) strongly in \( C(0, T; H^{m-\ell}(|x| < \rho)) \).

We apply this theorem with \( f_n \) replaced by \( v^n \), with \( \ell = 1 \) if \( \nu = 0 \) and \( \ell = 2 \) if \( \nu > 0 \); I will write the subsequence as \( v^n \) rather than \( v^{n_k} \). The PDE (14.1) tells us \( v^n \) is bounded in \( C(0, T_0; H^{m-\ell}(\mathbb{R}^3)) \), which we need for the Lipschitz hypothesis in the theorem. We conclude that for each \( \rho > 0 \) we have \( v^n \to v \) in \( C(0, T_0; H^{m-\ell}(|x| < \rho)) \). The limit \( v \) just obtained must be the same one as before, since each kind of convergence implies weak convergence in \( L^2(0, T_0; H^{m-\ell}(|x| < \rho)) \), and that limit must be unique. Now \( v \) is specified at each \( t \) in \( H^{m-\ell}_{loc} \); there is no ambiguity on a set of \( t \)'s of measure zero. Moreover, by (13.2), for any \( t \) there is a sequence \( t_j \to t \) with \( \|v(\cdot, t_j)\| \leq R_0 \). By Fact (6) above a subsequence converges weakly in \( H^m \), and the limit must be \( v(\cdot, t) \) because of the continuity of \( v \). We can now conclude that (13.2) holds for every \( t \), but we do not know yet that \( v \in C(0, T_0; H^m) \).

We need to check convergence for the pressure term. From the \( v^\varepsilon \) equation (10.1) we can write

\[
\nabla p^\varepsilon = -(I - P)S_\varepsilon(S_\varepsilon v^\varepsilon \cdot \nabla(S_\varepsilon v^\varepsilon)) \tag{13.4}
\]

Since \( v^n(\cdot, t) \) is bounded in \( H^m \), \( m \geq 3 \), we know from (10.11) and (7.4) that the right side is bounded in \( H^{m-1} \), uniformly in \( t \), and we can conclude as before that the sequence \( \nabla p^n \) corresponding to our current \( \varepsilon_n \) has a subsequence which converges weakly in \( L^2(0, T_0; H^{m-1}) \). (Once again we select a subsequence of a subsequence but do not change the notation.) We summarize what we know about \( v \) and the convergence:

\[
v \in L^\infty(0, T_0; H^m), \quad \|v(\cdot, t)\|_m \leq R_0, \quad 0 \leq t \leq T_0, \tag{13.5}
\]

\[
v \in C(0, T_0; H^{m-2}(|x| < \rho)), \quad \rho > 0, \tag{13.6}
\]

\[
v^n \to v \quad \text{weakly in} \quad L^2(0, T_0; H^m), \tag{13.7}
\]

\[
v^n \to v \quad \text{strongly in} \quad C(0, T_0; H^{m-2}(|x| < \rho)), \quad \rho > 0, \tag{13.8}
\]

\[
\nabla p^n \to \nabla p \quad \text{weakly in} \quad L^2(0, T_0; H^{m-1}). \tag{13.9}
\]

14 The limit solves the problem.

We have a sequence of solutions \( v^n, p^n \) of the regularized problem with \( \varepsilon = \varepsilon_n \to 0 \). We rewrite the problem (10.1), (10.2) as

\[
v^n_t + S_n(S_nv^n \cdot \nabla(S_nv^n)) + \nabla p^n = \nu S_n^2 \Delta v^n, \quad v^n(0) = S_nv_0 \tag{14.1}
\]

where of course \( S_n = S_{\varepsilon_n} \). We will make one more assumption: Assume the function \( \zeta \) in the definition (9.2) of \( S_\varepsilon \) is chosen so that \( \zeta(x) = 0 \) for \( |x| \geq 1 \). Then \( \zeta_\varepsilon(x) = 0 \) for \( |x| \geq \varepsilon \). We will always assume \( \varepsilon \leq 1 \). Thus \( (S_\varepsilon f)(x) \) depends on \( f(y) \) only for \( |y - x| \leq 1 \) at most. This will be convenient because we have strong convergence only on bounded sets.
We will verify that the limit \( v, p \) is an exact solution of the problem in Theorem 6.1. Since \( \nabla \cdot v^n = 0 \), the weak convergence (13.7) implies \( \nabla \cdot v \). We choose a vector-valued test function \( \phi \in C_0^\infty((0, T_0) \times \mathbb{R}^3; \mathbb{R}^3) \), take the scalar product with (14.1), and integrate in space-time. We integrate by parts in \( t \) in the first term. (We are assuming \( \phi = 0 \) at \( t = 0 \) and \( T_0 \).)

\[
- \int_0^{T_0} \int v^n \cdot \phi_t \, dx \, dt + \int_0^{T_0} \int [S_n v^n \cdot \nabla(S_n v^n)] \cdot S_n \phi \, dx \, dt + \int_0^{T_0} \int \nabla p^n \cdot \phi \, dx \, dt = \nu \int_0^{T_0} \int \Delta v^n \cdot S_n^2 \phi \, dx \, dt. \tag{14.2}
\]

We want to show the same identity holds with \( v^n \) replaced by \( v, p^n \) by \( p \), and \( S_n \) by \( I \). Since \( v_n \rightharpoonup v \) weakly in \( L^2(H^m) \) and therefore \( L^2(L^2) \),

\[
\int_0^{T_0} \int v^n \cdot \phi_t \, dx \, dt \rightharpoonup \int_0^{T_0} \int v \cdot \phi_t \, dx \, dt. \tag{14.3}
\]

Also \( \Delta v^n \rightharpoonup \Delta v \) in \( L^2(L^2) \) since \( v^n \rightharpoonup v \) in \( L^2(H^2) \), and \( S_n^2 \Delta \phi \rightharpoonup \Delta \phi \) strongly in \( L^2(L^2) \), so that by Fact 5 of §12,

\[
\int_0^{T_0} \int \Delta v^n \cdot S_n^2 \phi \, dx \, dt \rightharpoonup \int_0^{T_0} \int \Delta v \cdot \phi \, dx \, dt. \tag{14.4}
\]

The term with \( \nabla p^n \) term converges similarly.

We are left with the nonlinear term, and for this we use the strong convergence. Since \( \phi \) has compact support, there is some \( \rho > 1 \) so that \( \phi(x, t) = 0 \) for all \( (x, t) \) with \( |x| > \rho - 1 \). By the remark above, \( S_n \phi(x, t) = 0 \) for \( |x| > \rho \). We know that \( v^n \rightharpoonup v \) strongly in \( L^2((0, T_0); L^2(|x| < \rho + 2)) \). Since, for \( |x| < \rho \), \( S_n(v^n - v)(x) \) does not depend on \((v^n - v)(y, t)\) for \( |y| \geq \rho + 1 \), we can conclude from (9.5) that \( S_n(v^n - v) \rightharpoonup 0 \) strongly in \( L^2(0, T_0; L^2(|x| < \rho)) \). (To be careful, we could modify \((v^n - v)\) outside \( |x| > \rho + 1 \) to make it have compact support, and the modified function would converge strongly on \( L^2(0, T_0; L^2(\mathbb{R}^3)) \) so that (9.5) applies.) Also \( S_n v \rightharpoonup v \) strongly in \( L^2(L^2(\mathbb{R}^3)) \) and therefore in \( L^2(0, T_0; L^2(|x| < \rho)) \). Combining these two facts, we have

\[
\| S_n v^n - v \| \leq \| S_n (v^n - v) \| + \| S_n v - v \|,
\]

with norm in \( L^2(0, T_0; L^2(|x| < \rho)) \); each term on the right goes to zero, and thus

\[
S_n v^n \rightharpoonup v \quad \text{in} \quad L^2(0, T_0; L^2(|x| < \rho)). \tag{14.5}
\]

Now we are ready to show that the nonlinear term in (14.2) converges. Let \( w^n = S_n v^n \) and \( \psi^n = S_n \phi \). Integrating by parts (i.e., using (5.14)), we write this term as

\[
\int_0^{T_0} \int (w^n \cdot \nabla w^n) \cdot \psi^n \, dx \, dt = - \int_0^{T_0} \int w^n \cdot (w^n \cdot \nabla \psi^n) \, dx \, dt. \tag{14.6}
\]

As noted before, the integral is limited to \( |x| < \rho \). Since \( \nabla \phi \) is continuous with compact support, \( \nabla \psi^n = S_n(\nabla \phi) \rightharpoonup \nabla \phi \) uniformly. Using this and the \( L^2 \) convergence of \( w^n \) to \( v \) in
factors and the same and the third changing. In each case we estimate using $L^2$ norms for the two velocity factors and $L^\infty$ for the other. For instance

$$\left| \int \int w^n (v \cdot \nabla \phi) - \int \int v \cdot (v \cdot \nabla \phi) \right| \leq \int \int |(w^n - v) \cdot (v \cdot \nabla \phi)|$$

$$\leq \sup |\nabla \phi| \left( \int \int |w^n - v|^2 \right)^{1/2} \left( \int \int |v|^2 \right)^{1/2} \tag{14.7}$$

The middle factor goes to zero by (14.5) and the other two are bounded, and thus the product goes to zero. I suggest you give complete the argument as an exercise.

After passing to the limit in the nonlinear term we reverse the step in (14.6) and conclude

$$\int_0^{T_0} \int (-v \cdot \phi_t + (v \cdot \nabla v) \cdot \phi + \nabla p \cdot \phi - \nu \Delta v \cdot \phi) \, dxdt = 0 \tag{14.8}$$

and since $\phi$ is arbitrary this means that, as a distribution,

$$v_t = -v \cdot \nabla v - \nabla p + \nu \Delta v, \tag{14.9}$$

the correct differential equation. We can also check the initial condition easily.

15 Optimal regularity with $\nu = 0$.

We have almost proved Theorem 6.1. We have not yet shown that $v \in C(0, T_0; H^m(\mathbb{R}^3))$.

We do know $v \in L^\infty(0, T_0; H^m(\mathbb{R}^3))$. Also from the PDE (14.9) it is correct to say that $v(\cdot, t)$ is the $t$-integral of $v_t$, which is $L^\infty(H^{m-2})$, so that $v \in C(0, T_0; H^{m-2})$ at least. Moreover, Sobolev norms have the interpolation property

$$\|u\|_r \leq C\|u\|_0^{1-r/m}\|u\|_m^{r/m}, \quad 0 \leq r \leq m \tag{15.1}$$

and we can use this to check that $v \in C(0, T_0; H^r)$ for any $r < m$. To study the regularity further, it is best to consider the cases with and without viscosity separately. We start with $\nu = 0$, the case of the Euler equations.

Recall that $v(\cdot, t)$ is unambiguously defined, and according to (13.5) it is uniformly bounded in $H^m(\mathbb{R}^3)$. We first show that $v(\cdot, t)$ is weakly continuous in $H^m$, i.e., if $t_n \to t$, then $v(t_n) \rightharpoonup v(t)$ in $H^m$. Equivalently, we show that $D^\alpha v(t_n) \to D^\alpha v(t)$ in $L^2$ for each $\alpha$ with $|\alpha| \leq m$. We use the fact that $v \in C(0, T_0; H^{m-2})$. We need only consider $|\alpha| \geq m - 1$; let $\beta$ be some multi-index with $|\beta| = 2$ and $\beta \leq \alpha$. Given $\phi \in H^2(\mathbb{R}^3)$, $\langle D^\alpha v(t_n), \phi \rangle = \langle D^{\alpha-\beta} v(t_n), D^\beta \phi \rangle \to \langle D^{\alpha-\beta} v(t), D^\beta \phi \rangle = \langle D^\alpha v(t), \phi \rangle$. The weak convergence of $D^\alpha v(t_n)$ then follows from (13.5) and Fact 3 of §12.

Now according to Fact (2) of §12, to show $v \in C(0, T_0; H^m(\mathbb{R}^3))$, given the weak continuity, we only need to show $\|v(t)\|_m$ is continuous in $t$. To do this, we will multiply the equation for $v$ by $S_\varepsilon$, estimate the $t$-derivative of $\|S_\varepsilon v\|_m$, and then let $\varepsilon \to 0$. We will use a lemma stated below. Starting with the equation $S_\varepsilon v_t = -PS_\varepsilon (v \cdot \nabla v)$, we apply $D^\alpha$ with
$|\alpha| \leq m$ and take the inner product with $D^\alpha S_\varepsilon v$, resulting in the term $\langle D^\alpha S_\varepsilon (v \cdot \nabla v), D^\alpha S_\varepsilon v \rangle$ to be estimated. As usual we will add and subtract $\langle v \cdot \nabla (D^\alpha S_\varepsilon v), D^\alpha S_\varepsilon v \rangle = 0$. Because of $S_\varepsilon$ we need a further addition and subtraction. We write (interchanging some $D$’s and $S_\varepsilon$’s)

$$\frac{1}{2} \frac{d}{dt} \|D^\alpha S_\varepsilon v\|_0^2 = \langle S_\varepsilon D^\alpha (v \cdot \nabla v) - S_\varepsilon (v \cdot \nabla D^\alpha v), S_\varepsilon D^\alpha v \rangle$$

$$+ \langle S_\varepsilon (v \cdot \nabla D^\alpha v) - v \cdot S_\varepsilon (\nabla D^\alpha v), S_\varepsilon D^\alpha v \rangle. \quad (15.2)$$

The first term we estimate as usual since $S_\varepsilon$ is on the outside. For the second term we can use Lemma 15.1 below to get

$$\|S_\varepsilon (v \cdot \nabla D^\alpha v) - v \cdot S_\varepsilon (\nabla D^\alpha v)\|_0 \leq C\|\nabla v\|_{L^\infty} \|D^\alpha v\|_0. \quad (15.3)$$

Note there may be $m + 1$ derivatives on the left! Summing over $\alpha$ we have

$$\frac{d}{dt} \|S_\varepsilon v\|_m^2 \leq C\|\nabla v\|_{L^\infty} \|v\|_m^2. \quad (15.4)$$

The right side is bounded uniformly in $t$. If we integrate over $t_1 \leq t \leq t_2$, assuming $t_1 < t_2$, we get

$$\|S_\varepsilon v(t_2)\|_m^2 \leq \|S_\varepsilon v(t_1)\|_m^2 + C(t_2 - t_1) \leq \|v(t_1)\|_m^2 + C(t_2 - t_1) \quad (15.5)$$

and letting $\varepsilon \to 0$,

$$\|v(t_2)\|_m^2 \leq \|v(t_1)\|_m^2 + C(t_2 - t_1). \quad (15.6)$$

The estimate can be reversed in time (we could have put absolute value on the left in (15.4)), showing that $\|v(t)\|_m$ is continuous in $t$, thus completing our proof. Note this only works with $\nu = 0$!

**Lemma 15.1** Suppose $a : \mathbb{R}^n \to \mathbb{R}$ is $C^1$, with $a$ and $\nabla a$ uniformly bounded. Suppose $f \in L^2(\mathbb{R}^n)$. There is a constant $C$, independent of $a$ and $f$, so that for any first derivative $D$,

$$\|a(S_\varepsilon Df) - S_\varepsilon (aDf)\|_{L^2} \leq C\|\nabla a\|_{L^\infty} \|f\|_{L^2}. \quad (15.7)$$

Proof: (This is stated as a problem in M. Taylor’s book, p.3 of volume 3.) Notice that $Df$ may be worse than a function! We write out the convolution and integrate by parts:

$$\int [a(x) - a(y)] \zeta_\varepsilon(x - y) Df(y) \, dy$$

$$= \int Da(y) \zeta_\varepsilon(x - y) f(y) \, dy + \int [a(x) - a(y)] D\zeta_\varepsilon(x - y) f(y) \, dy. \quad (15.8)$$

We can estimate the first term as

$$\|\zeta_\varepsilon * ((Da)f)\|_{L^2} \leq \|\zeta_\varepsilon\|_{L^1} \|Da\|_{L^\infty} \|f\|_{L^2} \leq C\|\nabla a\|_{L^\infty} \|f\|_{L^2}. \quad (15.9)$$

For the second term, applying $D$ to $\zeta_\varepsilon$ produces a factor $\varepsilon^{-1}$, times $\varepsilon^{-n} D\zeta((x - y)/\varepsilon)$. We can write $a(x) - a(y) = b(x, y) \cdot (x - y)$ where $b$ is a vector-valued function bounded by $C\|\nabla a\|_{L^\infty}$. We can check (you check!) that $(z/\varepsilon)^{-n} D\zeta(z/\varepsilon)$, as a function of $z$, is bounded
in $L^1$ independent of $\varepsilon$. With this we estimate the second term in (15.8) as we would for a convolution of an $L^1$ function with an $L^2$ function. $\square$.

We note a further consequence of (15.4) describing the growth of $\|v(t)\|_m$ in time. We can integrate (15.4) from time 0 to $t$ and then let $\varepsilon \to 0$. Then $\|S_\varepsilon v(t)\|_m^2 \to \|v(t)\|_m^2$ and the same for time zero, so that

$$\|v(t)\|_m^2 \leq \|v(0)\|_m^2 + C \int_0^t \|\nabla v(s)\|_{L^\infty} \|v(s)\|_m^2 ds.$$  

(15.10)

It follows from Gronwall’s inequality that

$$\|v(t)\|_m^2 \leq \|v(0)\|_m^2 \exp \left( C \int_0^t \|\nabla v(s)\|_{L^\infty} ds \right).$$  

(15.11)

From this point the reader is well prepared to read the Beale-Kato-Majda paper, which says that if an initially smooth solution of the Euler equations loses smoothness, then the vorticity must become unbounded. Inequality (15.11) above is the same as (14) in that paper. Here is the reference:


16 Optimal regularity with $\nu > 0$.

This is less subtle than $\nu = 0$ because of the smoothing effect of viscosity. We write the Navier-Stokes equations as

$$v_t = \nu \Delta v + f, \quad f = -P(v \cdot \nabla v).$$  

(16.1)

Note for later use that

$$(v \cdot \nabla v)_i = \sum_j v_j v_{i,j} = \sum_j D_j(v_i v_j)$$  

(16.2)

or

$$v \cdot \nabla v = \sum_j D_j g_j, \quad (g_j)_i = v_i v_j.$$  

(16.3)

First consider the linear heat or diffusion equation

$$w_t = \nu \Delta w, \quad w(0) = w_0.$$  

(16.4)

The solution is naturally written in the Fourier transform as

$$\hat{w}(k, t) = e^{-\nu |k|^2 t} \hat{w}_0(k).$$  

(16.5)

We write this as

$$w = e^{\nu t \Delta} w_0$$  

(16.6)

and say that $\{e^{\nu t \Delta}, t \geq 0\}$ is the semigroup of operators generated by $\nu \Delta$. It is easy to see that $e^{\nu t \Delta}$ has norm $\leq 1$ on $L^2$ or $H^m$; also $e^{\nu t \Delta} w_0$ is continuous as a function of $t$ with values
in either space, as we can see by using the Lebesgue Dominated Convergence Theorem in the transform variable.

We consider (16.1) as a linear nonhomogeneous equation and write the solution as

\[ v(t) = e^{\nu t \Delta} v(0) + \int_0^t e^{\nu s \Delta} (-PD_j g_j(t - s)) \, ds. \]  

(16.7)

Here \( g_j \in L^\infty(0, T_0; H^m) \) and \( \|g_j(t)\|_m \leq C\|v\|_m^2 \), and it can be shown this equation holds at least in \( H^{m-1} \). Since \( \nu > 0 \) is fixed we will set \( \nu = 1 \) for simplicity. We commute operators and write (16.7) as

\[ v(t) = e^{t \Delta} v(0) - \int_0^t D_j e^{s \Delta} P g_j(t - s) \, ds. \]  

(16.8)

We will need to know that \( g_j \) is Hölder continuous in \( t \) with values in a space close to \( H^m \). We know that \( v \) is bounded in \( H^m \) and \( v_t \) is bounded \( H^{m-2} \) as remarked early in §15. Thus

\[ \|v(t_2) - v(t_1)\|_{m-2} \leq C|t_2 - t_1|. \]  

(16.9)

We choose \( \varepsilon \) with \( 0 < \varepsilon < 1 \). We check that \( v \) is Hölder continuous in \( t \) with values in \( H^{m-\varepsilon} \) using (16.9) and (15.1):

\[ \|v(t_2) - v(t_1)\|_{m-\varepsilon} \leq C\|v(t_2) - v(t_1)\|^{1-(\varepsilon/2)}_m \|v(t_2) - v(t_1)\|^{\varepsilon/2}_{m-2} \leq C'|t_2 - t_1|^{\varepsilon/2} \]  

(16.10)

It follows that \( g \) is also Hölder continuous in \( t \) with values in \( H^{m-\varepsilon} \).

The operator \( D_j e^{s \Delta} \) in (16.8) acts in the transform as multiplication by \( ik_j e^{-s|k|^2} \), and this is bounded by

\[ |k|e^{-s|k|^2} \leq (\sqrt{s})^{-1} \sqrt{s|k|^2} e^{s|k|^2} \leq C(\sqrt{s})^{-1}. \]  

(16.11)

We can conclude, for example, that on \( L^2 \), or \( H^m \), this operator has norm bounded by \( C/\sqrt{s} \).

It is important that we obtain an integrable power of \( s \). In a similar way we can show that \( D_j e^{s \Delta} : H^{m-\varepsilon} \rightarrow H^m \) has norm bounded by \( Cs^{-(1+\varepsilon)/2} \).

We can now check from (16.8) that \( v(t) \) is continuous in \( H^m \). The first term is continuous, as remarked below (16.6). For two times \( t_2 > t_1 \) the difference of the integral terms is

\[ \int_0^{t_1} D_j e^{s \Delta} P[g_j(t_2 - s) - g_j(t_1 - s)] \, ds + \int_{t_1}^{t_2} D_j e^{s \Delta} P g_j(t_2 - s) \, ds \equiv I_1 + I_2. \]  

(16.12)

For the second we estimate using the boundedness of \( g \) in \( H^m \) and (16.11),

\[ \|I_2\|_m \leq C \int_{t_1}^{t_2} s^{-1/2} \, ds = 2C(t_2^{1/2} - t_1^{1/2}) \leq 2C(t_2 - t_1)^{1/2}, \]  

(16.13)

For \( I_1 \) we use the Hölder condition on \( g \) in \( H^{m-\varepsilon} \).

\[ \|I_1\|_m \leq C \int_0^{t_1} s^{-(1+\varepsilon)/2} (t_2 - t_1)^{\varepsilon/2} \, ds \leq C'(t_2 - t_1)^{\varepsilon/2}. \]  

(16.14)

We have shown that the integral term in (16.8) is Hölder continuous in \( t \). This completes the proof that \( v \in C(0, T_0; H^m) \). We can actually show the solution is very smooth for \( t > 0 \).
Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^2$ boundary. We will prove an existence theorem for the Navier-Stokes equations

\[ v_t + v \cdot \nabla v + \nabla p = \nu \Delta v, \quad \nabla \cdot v = 0 \quad \text{in } \Omega \quad (17.1) \]

with the boundary condition

\[ v = 0 \quad \text{on } \partial \Omega \quad (17.2) \]

and initial condition

\[ v(\cdot, 0) = v_0, \quad v_0 \in H_0^1(\Omega), \quad \nabla \cdot v_0 = 0. \quad (17.3) \]

Note that if $w : \bar{\Omega} \to \mathbb{R}^3$ and $q : \bar{\Omega} \to \mathbb{R}$ are smooth then

\[ \int_{\Omega} q \nabla \cdot w + \int_{\Omega} w \cdot \nabla q = \int_{\partial \Omega} q w \cdot n \, dS. \quad (17.4) \]

If $w$ is fixed but $q$ is arbitrary, then

\[ \langle w, \nabla q \rangle_{L^2} = 0 \quad \forall q \quad \text{iff} \quad \nabla \cdot w = 0 \text{ in } \Omega \quad \text{and} \quad w \cdot n = 0 \text{ on } \partial \Omega. \quad (17.5) \]

This fact suggests a natural way to project onto divergence vector fields in $L^2(\Omega)$. Let

\[ G = \{ \nabla q \in L^2(\Omega) : q \in H^1(\Omega) \} \quad (17.6) \]

and let $P$ be the orthogonal projection onto the complement $G^\perp$. The identity (17.4) holds for $q, w \in H^1$; recall that boundary values of $H^1$ functions are meaningful. Thus for $w \in H^1$, $w \in PL^2$ iff $\nabla \cdot w = 0$ in $\Omega$ and $w \cdot n = 0$ on $\partial \Omega$. For general $w \in PL^2$ we can say $\nabla \cdot w = 0$ as a distribution; we also have that $w \cdot n = 0$ on $\partial \Omega$ in a weak sense, even though we could not ordinarily talk about the restriction of $v$ to the boundary.

Later we will use the fact that $P$ is a bounded linear operator on $H^1(\Omega; \mathbb{R}^3)$,

\[ \|Pw\|_{H^1} \leq C\|w\|_{H^1}. \quad (17.7) \]

Given $w \in H^1$, $(I - P)w$ has the form $(I - P)w = \nabla q$, where $\Delta q = \nabla \cdot w$ and $\partial q / \partial n = w \cdot n$ on $\partial \Omega$. This is a Neumann problem for $q$; a necessary compatibility condition is satisfied, and we choose the constant in $q$ so that $\int_{\Omega} q = 0$. From the sharp regularity theorem for elliptic pde’s in Sobolev spaces we can conclude that $q \in H^2$ and $\|q\|_{H^2} \leq C\|w\|_{H^1}$, which implies (17.7).

We will work with special Sobolev spaces for velocity fields

\[ V^0 = PL^2(\Omega; \mathbb{R}^3) \quad (17.8) \]

\[ V^1 = H_0^1(\Omega; \mathbb{R}^3) \cap V^0 = \{ v \in H^1 : \nabla \cdot v = 0 \text{ in } \Omega, \ v = 0 \text{ on } \partial \Omega \} \quad (17.9) \]

\[ V^2 = H^2(\Omega; \mathbb{R}^3) \cap V^1 = \{ v \in H^2 : \nabla \cdot v = 0 \text{ in } \Omega, \ v = 0 \text{ on } \partial \Omega \} \quad (17.10) \]
For a good solution of (17.1–2) we can obtain the same energy estimate as before,

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 = -\nu \int_{\Omega} |\nabla v|^2. \] (17.11)

When we integrate by parts there are boundary terms to consider, but they are zero because \( v = 0 \) on \( \partial \Omega \). This estimate will be fundamental. There is an important existence theorem for a “weak” solution, for all time, in the class suggested by the estimate, \( L^\infty(0; T; V^0) \cap L^2(0, T; V^1) \). However these weak solutions, properly defined, are weak enough that we can’t prove they are unique. We will prove here that a better or “strong” solution exists for a short time, or for all time if the initial state is small enough in a certain sense, and that this solution is unique.

We will write the evolution equation in the form

\[ v_t + P(v \cdot \nabla v) = \nu P\Delta v. \] (17.12)

Note that \( P\Delta v \neq \Delta v \) because \( \Delta v \cdot n \neq 0 \) on \( \partial \Omega \) in general. \( P \) and derivatives do NOT commute as they did in free space.

18 The Stokes operator.

We will interpret \(-P\Delta\) as an unbounded operator \( A \) with dense domain on \( V^0 \), with zero boundary condition, having an inverse which is bounded on \( V^0 \). Note that if we assume \( u \in V^1 \), \(-P\Delta u \in V^0 \), and \( w \in V^1 \), then

\[ \langle -P\Delta u, w \rangle_{L^2} = \langle -\Delta u, Pw \rangle_{L^2} = \langle -\Delta u, w \rangle_{L^2} = \langle \nabla u, \nabla w \rangle_{L^2}. \] (18.1)

We can define an inner product on \( H^1_0 \), and thus on \( V^1 \), as

\[ \langle u, w \rangle_{H^1_0} = \langle \nabla u, \nabla w \rangle_{L^2}. \] (18.2)

This is sensible because of the Poincare inequality, which says that in \( H^1_0 \) the \( L^2 \) norm is bounded by the norm from this inner product. From now on we will write \( \langle \cdot, \cdot \rangle_0 \) for the \( L^2 \) inner product and \( \langle \cdot, \cdot \rangle_1 \) for the \( H^1_0 \) inner product.

Let \( Au = -P\Delta u \). We can regard \( A \) as an unbounded operator on \( V^0 \) with (dense) domain \( V^2 \). We will see that it is a closed operator. From (18.1) we have, for \( u \in V^2 \) and \( w \in V^1 \),

\[ \langle Au, w \rangle_0 = \langle u, w \rangle_1. \] (18.3)

From this and usual Hilbert space arguments, we can show that, given \( f \in V^0 \), there is a unique \( u \in V^1 \), with \( \|u\|_1 \leq C\|f\|_0 \), so that \( u \) is a “weak” solution of \( Au = f \), that is,

\[ \langle u, w \rangle_1 = \langle f, w \rangle_0 \ \forall w \in V^1. \] (18.4)

The operator \( R : f \mapsto u \) is bounded \( V^0 \to V^1 \). It follows from the Rellich Compactness Theorem that it is compact as a bounded operator from \( V^0 \) to \( V^0 \). We can check using (18.3) that it is self-adjoint. The important consequences about its spectrum will be used below. As for scalar elliptic operators such as \( \Delta \), we can prove, with some work, that the “weak”
solution $u \in V^1$ is actually in $V^2$, depending boundedly on $f \in V^0$, that is, $Au = f$ in $V^0$ and $\|u\|_2 \leq C\|f\|_0$. Such a regularity result is one of the main conclusions of the elliptic pde course, usually for scalar pde’s; in this case we have a system. Thus we are justified in saying that $R$ is $A^{-1}$. As a consequence of this fact, $A$, as an unbounded operator on $V^0$ with domain $V^2$, is closed. For later use we restate the estimate above:

$$\|u\|_{H^2} \leq C\|P\Delta u\|_{L^2} \quad \text{for} \quad u \in V^2. \quad (18.5)$$

We noted that $A^{-1} : V^0 \to V^0$ is self-adjoint and compact. From a key theorem in functional analysis, we know that there is a complete orthonormal sequence of eigenfunctions $\psi_n$ in $V^0$. That is, $A^{-1}\psi_n = \lambda_n^{-1}\psi_n$, or $A\psi_n = \lambda_n\psi_n$ for some $\lambda_n$. Necessarily $\lambda_n > 0$ for each $n$ and $\lambda_n \to \infty$. From the remarks above we know $\psi_n \in V^2$. Since they are orthonormal in $V^0$, for any $f \in V^0$ we can write

$$f = \sum_{n=1}^{\infty} \langle f, \psi_n \rangle_0 \psi_n \quad (18.6)$$

(sum converging in $V^0$). However, the $\psi_n$ are orthogonal in $V^1$ also:

$$\langle \psi_n, \psi_m \rangle_1 = \langle \nabla \psi_n, \nabla \psi_m \rangle_0 = \langle -\Delta \psi_n, \psi_m \rangle_0 = \langle -\Delta \psi_n, P\psi_m \rangle_0$$

$$= \langle -P\Delta \psi_n, \psi_m \rangle_0 = \langle A\psi_n, \psi_m \rangle_0 = \lambda_n \langle \psi_n, \psi_m \rangle_0 = \lambda_n \delta_{mn}. \quad (18.7)$$

We can check that $\left\{\lambda_n^{-1/2}\psi_n\right\}$ are a complete orthonormal system in $V^1$. Also from (18.4), $\langle w, \psi_n \rangle_1 = \lambda_n \langle w, \psi_n \rangle_0$. It follows that for any $w \in V^1$

$$w = \sum_{n=1}^{\infty} \langle w, \psi_n \rangle_0 \psi_n \quad (18.8)$$

(sum converging in $V^1$).

19 The Galerkin approximation.

To prove the existence of exact solutions for the Navier-Stokes equations we will use the Galerkin method; that is, we use an approximation in a finite dimensional subspace. (This approach is closely related to numerical methods such as the finite element method.)

Let $P_n$ be the orthogonal projection in $V^0 = PL^2$ on the span of $\psi_1, \ldots, \psi_m$,

$$P_n f = \sum_{j=1}^{n} \langle f, \psi_j \rangle_0 \psi_j. \quad (19.1)$$

Then for $f \in V^0$, $P_n f \to f$ in $V^0$ as $n \to \infty$, by (18.6). Also, for $f \in V^1$, $P_n f \to f$ in $V^1$ as $n \to \infty$, by (18.8). Now, if we have an exact solution of (17.12), we can apply $P_n$ to get

$$P_n v_t + P_n (v \cdot \nabla v) = \nu P_n (\Delta v). \quad (19.2)$$

$P_n v$ depends on $\langle v, \psi_j \rangle_0$ only for $j \leq n$, but $P_n (v \cdot \nabla v)$ depends on higher $\psi$’s as well. Our finite-dimensional approximation will have to solve a different equation.
We will seek an approximate solution $v^n$ in $V^2$ of the form
\[ v^n(x, t) = \sum_{j=1}^{n} a_j(t) \psi_j(x) \] (19.3)
with coefficients $a_j(t)$ to be found so that
\[ v^n_t + P_n(v^n \cdot \nabla v^n) = \nu P\Delta v^n, \quad v^n(\cdot, 0) = P_n v_0. \] (19.4)
(Note that $P_n \Delta v^n = P\Delta v^n$ because the $\psi_j$ are eigenfunctions of $P\Delta$. The coefficients $a_j$ depend on $n$, but we are not indicating that.) This is the Galerkin approximation.

Taking $\langle \cdot, \psi_k \rangle_0$ through the equation (19.4), we can reduce the problem to a system of $n$ ordinary differential equations (with $t = d/dt$)
\[ a_k'(t) + \sum_{j, \ell=1}^{n} b_{j\ell k} a_j a_\ell = -\nu \lambda_k a_k, \quad 1 \leq k \leq n, \] (19.5)
\[ a_k(0) = \langle v_0, \psi_k \rangle_0, \quad b_{j\ell k} = \langle \psi_j \cdot \nabla \psi_\ell, \psi_k \rangle, \] (19.6)
The ODE system has a solution at least for a short time. Since it is nonlinear, we cannot say from general theory that solutions exist for all time, but in fact we can show that from an energy estimate. We multiply (19.4) by $v^n$, integrate in $x$ and find as usual that
\[ \frac{1}{2} \frac{d}{dt} \| v^n \|_{L^2}^2 + \nu \| \nabla v^n \|_{L^2}^2 = 0. \] (19.7)
It is important here that
\[ \langle P_n(v^n \cdot \nabla v^n), v^n \rangle = \langle P(v^n \cdot \nabla v^n), P_n v^n \rangle = \langle (v^n \cdot \nabla v^n), v^n \rangle = 0. \] (19.8)
It follows that
\[ \sum_{j=1}^{n} a_j(t)^2 = \| v^n(t) \|_{L^2}^2 \leq \| v^n(0) \|_{L^2}^2 = \sum_{j=1}^{n} a_j(0)^2. \] (19.9)
Thus with $n$ fixed the $a_j$’s remain bounded. It follows from ODE theory that the solution can be continued for all time. Also it follows from (19.7) that
\[ \frac{1}{2} \| v^n(t) \|_{L^2}^2 + \nu \int_0^t \| \nabla v^n(s) \|_{L^2}^2 ds = \frac{1}{2} \| v^n(0) \|_{L^2}^2 \leq \frac{1}{2} \| v_0 \|_{L^2}^2. \] (19.10)
Thus for any time $T$ the sequence $\{ v^n \}$ is bounded in the space $L^{\infty}(0, T; V^0) \cap L^2(0, T; V^1)$.

We will derive a higher estimate for $v^n$ using (19.4). We multiply by $\Delta v^n$ and integrate in $x$, integrating by parts in the first term, to get
\[ \frac{1}{2} \frac{d}{dt} \| \nabla v^n \|_{L^2}^2 + \nu \| P\Delta v^n \|_{L^2}^2 = \int_{\Omega} P_n(v^n \cdot \nabla v^n) \cdot \Delta v^n \, dx. \] (19.11)
In the last term we can replace $\Delta$ by $P\Delta$ without change, but then $P_n$ can be omitted, since $P_n P\Delta v^n = P\Delta v^n$. We estimate using (3.1),
\[ \left| \int_{\Omega} (v^n \cdot \nabla v^n) \cdot P\Delta v^n \, dx \right| \leq \| v^n \|_{L^4} \| \nabla v^n \|_{L^4} \| P\Delta v^n \|_{L^2}. \] (19.12)
Also for any \( f \in H^1(\Omega) \),

\[
\|f\|_{L^4} \leq \|f\|_{L^6}^{3/4} \|f\|_{L^2}^{1/4} \leq C \|f\|_{H^1}^{3/4} \|f\|_{L^2}^{1/4} \tag{19.13}
\]

where we have used (3.3) and then a Sobolev inequality as in (4.3) but for a bounded domain. We apply this to \( v^n \). Since \( v^n = 0 \) on \( \partial \Omega \), we can use Poincaré’s inequality in the first factor and obtain

\[
\|v^n\|_{L^4} \leq C \|\nabla v^n\|_{L^2}^{3/4} \|v^n\|_{L^2}^{1/4}. \tag{19.14}
\]

Also we can apply (19.13) to \( \nabla v^n \) and use (18.5) to get

\[
\|\nabla v^n\|_{L^4} \leq C \|v^n\|_{H^2}^{3/4} \|\nabla v^n\|_{L^2}^{1/4} \leq C' \|P\Delta v^n\|_{L^2}^{3/4} \|\nabla v^n\|_{L^2}^{1/4}. \tag{19.15}
\]

Now we combine (19.14–15) with (19.12) to estimate

\[
\left| \int_{\Omega} (v^n \cdot \nabla v^n) \cdot P\Delta v^n \, dx \right| \leq C \|P\Delta v^n\|_{L^2}^{7/4} \|\nabla v^n\|_{L^2} \|v^n\|_{L^2}^{1/4} \leq C' \|P\Delta v^n\|_{L^2} \|\nabla v^n\|_{L^2} \tag{19.16}
\]

since \( \|v^n\|_{L^2} \leq \|v_0\|_{L^2} \), a constant. Next we use Young’s inequality

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{19.17}
\]

to separate the factors on the right in (19.16). With \( p = 8/7, q = 8 \), we get

\[
\left| \int_{\Omega} (v^n \cdot \nabla v^n) \cdot P\Delta v^n \, dx \right| \leq \varepsilon \|P\Delta v^n\|_{L^2}^2 + C_\varepsilon \|\nabla v^n\|_{L^2}^8 \tag{19.18}
\]

with \( \varepsilon \) to be chosen. Combining this with (19.11) gives

\[
\frac{1}{2} \frac{d}{dt} \|\nabla v^n\|_{L^2}^2 + \nu \|P\Delta v^n\|_{L^2}^2 \leq \varepsilon \|P\Delta v^n\|_{L^2}^2 + C_\varepsilon \|\nabla v^n\|_{L^2}^8. \tag{19.19}
\]

We choose \( \varepsilon = \nu/2 \) and simplify to get

\[
\frac{d}{dt} \|\nabla v^n\|_{L^2}^2 + \nu \|P\Delta v^n\|_{L^2}^2 \leq C_1 \|\nabla v^n\|_{L^2}^8. \tag{19.20}
\]

To handle the term on the right, we estimate

\[
\|\nabla v^n\|_{L^2} \leq \|v^n\|_{H^1} \leq C \|v^n\|_{H^2}^{1/2} \|v^n\|_{L^2}^{1/2} \leq C' \|P\Delta v^n\|_{L^2}^{1/2} \tag{19.21}
\]

using (18.5) once again, and the boundedness of \( v^n \) in \( L^2 \), or

\[
\|P\Delta v^n\|_{L^2}^2 \geq c_0 \|\nabla v^n\|_{L^2}^4 \tag{19.22}
\]

for some \( c_0 > 0 \). Applying this to (19.20) we get

\[
\frac{d}{dt} \|\nabla v^n\|_{L^2}^2 \leq C_1 \|\nabla v^n\|_{L^2}^8 - c_0 \nu \|\nabla v^n\|_{L^2}^4. \tag{19.23}
\]
Letting \( y(t) = \|\nabla v^n(t)\|_{L^2}^2 \), we now have the differential inequality
\[
y' \leq C_1 y^4 - c_0 \nu y^2 \quad \text{or} \quad y' \leq C_1 y^2 (y^2 - c_0 \nu / C_1) \tag{19.24}
\]
The right side is \(< 0\) for \( y < y_0 \) and \( > 0\) for \( y > y_0 \), where \( y_0 = \sqrt{c_0 \nu / C_1} \). If \( y(0) < y_0 \), we can argue that \( y(t) \) remains bounded by \( y_0 \) for all time. If not, it is bounded for some sufficiently short time \( t \leq T \). Returning to (19.20) and integrating in \( t \), we can conclude that there exist \( T \) and \( R_0 \), independent of \( n \), but depending on \( \|\nabla v(0)\|_{L^2}^2 \), so that for \( t \leq T \),
\[
\|\nabla v^n(t)\|_{L^2}^2 + \nu \int_0^t \|\Delta v^n(s)\|_{L^2}^2 \, ds \leq R_0 \tag{19.25}
\]
In view of (18.5) we can say in summary that
\( v^n \) is bounded in \( L^\infty(0, T; V^1) \cap L^2(0, T; V^2) \).
\( \tag{19.26} \)

We will also show that
\( v^n_t \) is bounded in \( L^2(0, T; V^0) \)
\( \tag{19.27} \)
using (19.4). The boundedness follows for the viscosity term from (19.25). For the nonlinear term we estimate at one time
\[
\|v^n \cdot \nabla v^n\|_{L^2} \leq C\|v^n\|_{L^\infty} \|\nabla v^n\|_{L^2} \leq C'\|v^n\|_{H^2} \|\nabla v^n\|_{L^2} \leq C''\|v^n\|_{H^2} \tag{19.28}
\]
where we have used (19.25). Thus
\[
\int_0^T \|v^n \cdot \nabla v^n\|_{L^2}^2 \, dt \leq C \int_0^T \|v^n\|_{H^2}^2 \, dt \tag{19.29}
\]
and we have already noted that this quantity is bounded.

20 The existence proof.

We are ready for the theorem of existence and uniqueness.

**Theorem 20.1** For the initial value problem (17.1–3), given \( v_0 \in V^1 \), there is a solution for \( 0 \leq t \leq T \) with \( v \in L^2(0, T; V^2) \), \( v \in L^\infty(0, T; V^1) \), \( v_t \in L^2(0, T; V^0) \), where \( T \) depends on \( \|v_0\|_{H^1} \). If \( \|v_0\|_{H^1} \) is small enough, the solution can be continued for all time. The solution is unique in this class.

Since \( v \) and \( v_t \) are both in \( L^2(0, T; V^0) \), we can regard \( v(t) \) as a continuous function in \( V^0 \). Thus it is meaningful to say that \( v(0) = v_0 \).

**Proof of existence.** From (19.26) we can conclude that a subsequence of \( v^n \) converges weakly in \( L^2(0, T; V^2) \) to a limit \( v \). We can also assume using (19.27) that \( v^n_t \) converges weakly in \( L^2(0, T; L^2) \); the limit is \( v_t \). Further, we know from (19.26–27) that \( v^n, \nabla v^n, v^n_t \) are all bounded in \( L^2(0, T; L^2) \equiv L^2(\Omega \times (0, T)) \). That is, \( v^n \) is bounded in \( H^1(\Omega \times (0, T)) \). From the Rellich Compactness Theorem, a further subsequence converges strongly in \( L^2(\Omega \times (0, T)) \).
Since \( v^n \) is bounded in \( L^2(0, T; V^2) \) and for each \( t \), \( \| v^n(0) \|_{H^1} \leq \| v^n(t) \|_{H^2}^{1/2} \| v^n(t) \|_{L^2}^{1/2} \), we also have
\[
v^n \to v \quad \text{strongly in} \quad L^2(0, T; V^1) .
\] (20.1)

Since \( v^n(t) \) is uniformly bounded in \( V^1 \), we also have \( v \in L^\infty(0, T; H^1) \) by the same measure-theory argument as before. Since the \( L^2 \) norm is bounded by the \( H^1 \) norm, as in (19.13), we note for later use that
\[
\begin{align*}
v^n, v & \text{ are bounded in } L^\infty(0, T; L^4) , \quad (20.2) \\
v^n \to v & \text{ strongly in } L^2(0, T; L^4) . \quad (20.3)
\end{align*}
\]

We now argue that \( v \) solves the right equation. We will multiply by a test function and integrate. Suppose \( \phi \in C^1(0, T; H^1) \). For each \( t \), \( P_n\phi(\cdot, t) \to P\phi(\cdot, t) \) in \( H^1 \), and by the Lebesgue Convergence Theorem, \( P_n\phi \to P\phi \) in \( L^2(0, T; H^1) \). In particular
\[
\nabla P_n\phi \to \nabla P\phi \quad \text{in} \quad L^2(0, T; L^2) . \quad (20.4)
\]

We multiply (19.4) by \( \phi \) and integrate in \((x, t)\) to get
\[
\int_0^T \int_0^T v^n_t \phi + \int_0^T \int (v^n \cdot \nabla v^n) \cdot P_n\phi = \nu \int_0^T \int P\Delta v^n \phi . \quad (20.5)
\]

Since \( v^n_t \to v_t \) in \( L^2(0, T; L^2) \), the first integral converges to \( \int \int v_t \phi \). Since \( v^n \to v \) in \( L^2(0, T; V^2) \), and thus \( P\Delta v^n \to P\Delta v \) in \( L^2(0, T; L^2) \), the last integral converges to \( \int \int P\Delta v \phi \). Moving \( v^n \cdot \nabla \), we need to show that
\[
-\int \int v^n \cdot (v^n \cdot \nabla) P_n\phi \to -\int \int v \cdot (v \cdot \nabla) P\phi = \int \int P(v \cdot \nabla v) \cdot \phi . \quad (20.6)
\]

We do this using (20.2,3,4). We add and subtract, in abbreviated notation, as
\[
v^n v^n \phi - vv \phi = v^n v^n (\phi - \phi) + v^n (v^n - v) \phi + (v^n - v)vv . \quad (20.7)
\]

For the middle term, for instance, we estimate
\[
\begin{align*}
\left| \int_0^T \int_t \int (v^n - v) \cdot \nabla P\phi \right| dt & \leq \int_0^T \| v^n \|_{L^4} \| v^n - v \|_{L^4} \| \nabla P\phi \|_{L^2} dt \\
& \leq \| v^n \|_{L^\infty(L^1)} \| \phi \|_{L^\infty(H^1)} \int_0^T \| v^n - v \|_{L^4} dt \leq T^{1/2} \| v^n \|_{L^\infty(L^4)} \| \phi \|_{L^\infty(H^1)} \| v^n - v \|_{L^2(L^4)}
\end{align*}
\] (20.8)

using the Cauchy-Schwarz inequality in the last step. The last expression goes to zero by (20.4) Having justified the limits of individual terms, we now have in the limit
\[
\int_0^T \int_0^T (v_t \phi) + \int_0^T \int (Pv \cdot \nabla v) \cdot \phi = \nu \int_0^T \int P\Delta v \phi \quad (20.9)
\]

and since \( \phi \) is arbitrary, the expected equation holds,
\[
v_t + P(v \cdot \nabla v) = \nu P\Delta v . \quad (20.10)
\]

27
To complete the proof of existence, we need to show that \( v(0) = v_0 \), the prescribed initial state. For any \( \phi \in C^1(0, T; V^0) \) we have
\[
\int_0^T \langle v^n_t, \phi \rangle_0 \, dt + \int_0^T \langle v^n, \phi \rangle_0 \, dt = \langle v^n(T), \phi(T) \rangle_0 - \langle v^n(0), \phi(0) \rangle_0 \tag{20.11}
\]
and similarly for \( v \), thinking of it as an element of \( H^1(0, T; V^0) \). As \( n \to \infty \), the two terms on the left with \( v^n \) converge to those with \( v \). We choose \( \phi(x, t) = (T - t) \psi_k(x) \) for any \( k \).
Looking at the right side of the equation, we conclude that \( -T \langle v^n(0), \psi_k \rangle_0 \to -T \langle v(0), \psi_k \rangle_0 \).
But \( \langle v^n(0), \psi_k \rangle_0 \to \langle v_0, \psi_k \rangle_0 \), since \( v^n(0) = P_n v_0 \). Thus \( \langle v(0), \psi_k \rangle_0 = \langle v_0, \psi_k \rangle_0 \) for each \( k \), and we conclude that \( v(0) = v_0 \) since both are in \( V_0 \).

21 The uniqueness proof.

Suppose \( v_1 \) and \( v_2 \) are two solutions of the same problem (17.1–3) in the class specified in Theorem 20.1. Let \( w = v_1 - v_2 \). We prove that \( w = 0 \) by obtaining an energy estimate. Subtracting the two equations we have
\[
w_t + P(v_1 \cdot \nabla w) + P(w \cdot \nabla v_2) = \nu P \Delta w, \tag{21.1}
\]
an equation that looks similar to Navier-Stokes. Multiply by \( w \) and integrate in \( x \) and then \( t \) to get, noting that \( w(\cdot, 0) = 0 \),
\[
\frac{1}{2} \| w(\cdot, t) \|_{L^2}^2 + \nu \int_0^t \| \nabla w \|_{L^2}^2 \leq \int_0^t \langle v_1 \cdot \nabla w, w \rangle \, dt + \int_0^t \langle w \cdot \nabla v_2, w \rangle \, dt. \tag{21.2}
\]
The first integral on the right is zero as usual. For the second we integrate by parts and estimate
\[
|\langle w \cdot \nabla v_2, w \rangle| = |\langle v_2, w \cdot \nabla w \rangle| \leq \| v_2 \|_{L^\infty} \| w \|_{L^2} \| \nabla w \|_{L^2} \leq (\nu/2) \| \nabla w \|_{L^2}^2 + C_\nu \| v_2 \|_{H^2}^2 \| w \|_{L^2}^2 \tag{21.3}
\]
using \( ab \leq \varepsilon a^2 + C\varepsilon b^2 \). Putting (21.3) in (21.2) we have
\[
\frac{1}{2} \| w(\cdot, t) \|_{L^2}^2 + \nu \int_0^t \| \nabla w \|_{L^2}^2 \leq \frac{\nu}{2} \int_0^t \| \nabla w \|_{L^2}^2 \, dt + C_\nu \int_0^t \| v_2 \|_{H^2}^2 \| w \|_{L^2}^2 \, dt \tag{21.4}
\]
and combining and then neglecting the \( \nu \) terms
\[
\| w(t) \|_{L^2}^2 \leq C \int_0^t \| v_2(s) \|_{H^2}^2 \| w(s) \|_{L^2}^2 \, dt. \tag{21.5}
\]
We are now in the situation of Gronwall’s inequality. Let \( z(t) \) be the integral on the right. Then \( z'(t) \leq g(t) z(t) \) where \( g(t) = C \| v_2(t) \|_{H^2}^2 \). We know that \( g \) is integrable since \( v_2 \in L^2(H^2) \). We can conclude \( z(t) \leq z(0) \exp \int_0^t g(s) \, ds \). But \( z(0) = 0 \) and therefore \( z(t) \equiv 0 \), that is \( w \equiv 0 \).
Notice that in the uniqueness proof we did not use much information about $v_1$. That was because of the fact that the nonlinearity is bilinear. We can prove that if the problem has a “strong” solution and a “weak” solution on the same time interval, then they must be the same. However, we can’t prove that two “weak” solutions are the same.

If a solution which is smooth for a short time eventually loses regularity, there is still a “weak” solution, but we don’t know whether it is unique. Whether or not regularity is actually lost is not known and is the important unsolved problem, one of the Clay Mathematics Institute’s millennium problems. (We could say that the uniqueness of the weak solution is a closely related unsolved problem.) There are many results that say solutions are “almost” regular. A famous theorem from a few years ago, says that a weak solution must be regular except on a set in space-time which is so small that its one-dimensional Hausdorff measure is zero, but it could still be a dense set. This was proved by Caffarelli, R. Kohn, and L. Nirenberg, improving on earlier work by V. Scheffer. Leray in his pioneering work proved that for a given weak solution at most times (except for a small set) the solution is regular in space. There are theorems that for most initial states the solution is regular. It is not hard to see, by an estimate similar to that in Sec. 20, that if we somehow know that the $L^4$ norm of $v(t)$ is bounded independent of $t$ as long as the solution lasts, then we can get a “strong” solution for all time. Once we have a that, we can prove that a smooth solution remains smooth. So the gap in regularity is at a low level, going from $L^2$ to $L^4$. (This remark can be improved with much work to $L^3$ rather than $L^4$.)

There is a brief description of what is known and not known by C. Fefferman at the Clay website, www.claymath.org/millennium/Navier-Stokes_Equations, including some references. There is also a video of a lecture by L. Caffarelli.

Another reference is a set of lecture notes by C. Constantin and C. Foias on the Navier-Stokes equations, published by the U. Chicago press, in the style of a graduate course. There are two books by R. Temam on the Navier-Stokes equations, one large and comprehensive, and the other a small book published by SIAM. Once again I will recommend Michael Taylor’s three-volume book on PDE and the book of Majda and Bertozzi, Vorticity and Incompressible Flow. The first part of these notes, existence for Euler and Navier-Stokes in free space, is much like Chapter 3 of Majda and Bertozzi and Chapter 17 of Taylor, Vol. 3.