Conservation of energy and compressible Euler equations

If the fluid is compressible, we have to find other equations to close the system.

We now consider the energy consisting of kinetic energy and internal energy

\[ E = \frac{1}{2} \rho |u|^2 + \rho e. \]

where \( e \) is the specific internal energy (internal energy per unit mass).

Consider a moving fluid element with volume \( R_t \). By the convection theorem, we have

\[
\frac{d}{dt} \int_{R_t} E \, dV = \int_{R_t} E_t + \nabla \cdot (Eu) \, dV.
\]

What can change the energy inside this bulk? By the first principle of thermodynamics, both heat transfer and work can change the total energy.

Since we are doing ideal gas, there is no heat transfer, because heat transfer usually results in viscosity.

Then, we have to compute the work done on the fluid element. The work rate done by pressure on the surface is

\[
\int_{\partial R_t} -pudS
\]

The work done by the body force is

\[
\int_{R_t} \rho b \cdot udV
\]

Hence, conservation of energy reads

\[ E_t + \nabla \cdot ((E + p)u) = \rho b \cdot u \]

The compressible Euler equations read

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho uu + pI) &= b, \\
E_t + \nabla \cdot ((E + p)u) &= \rho b \cdot u
\end{align*}
\]

We have one more equation, but with one more unknowns. We have to find the equation of state to complete them.
Review of ideal gas law

For ideal gas, the ideal gas law reads

\[ pV = nRT. \]

By the first law of thermodynamics,

\[ dU = \bar{d}Q - pdV. \]

By well-known,

\[ dU = nC_vdT \]

We assume the ideal gas is adiabatic, and thus

\[ -pdV = nC_vdT = nC_v \frac{Vdp + pdV}{nR} \]

Hence,

\[ -(\frac{R}{C_v} + 1) \frac{dV}{V} = \frac{dp}{p} \]

Letting \( \gamma = 1 + \frac{R}{C_v} \), we have

\[ p = CV^{-\gamma} = C' \rho^\gamma. \]

The enthalpy for ideal gas is defined as

\[ H = U + pV = n(C_v + R)T. \]

Note that if the gas is expanded with constant pressure, by the conservation of energy, we have

\[ nC_vdT = nC_pdT - pdV = (nC_p - nR)dT \]

Hence, \( C_v + R = C_p \).

We check that (since \( \bar{d}Q = 0 \))

\[ dH = Vdp \propto \frac{1}{\rho}dp. \]

Note that \( \bar{d}Q = TdS \). Hence, for adiabatic process, the entropy \( S \) is a constant.
Gas dynamics

Above we assume the system has uniform properties throughout the whole volume $V$. Now, imagine, we have a distribution of gases, the pressure, etc may not be uniform. We can imagine the gas as many adiabatic blocks.

We have similarly the following equations for each $x$ and $t$ ($c_v, c_p$ are now specific heat capacities):

\begin{align*}
  p &= R \rho T, \\
  e &= c_v T = \frac{c_v}{R} \frac{p}{\rho}, \\
  h &= e + p/\rho = c_p T
\end{align*}

(0.2)

As a result, we find $\rho e = \frac{c_v}{R} p = \frac{p}{\gamma - 1}$. Hence, the energy is given by

\[ E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho |u|^2. \]

This is the equation of state which closes the system.

Isentropic fluids

Since the fluid elements here are adiabatic, then the entropy is constant for each parcel. The specific entropy (entropy per unit mass) is then constant along fluid particles. Hence,

\[ s_t + u \cdot \nabla s = 0. \]

This equation can be derived from first principles, but we do not do this here.

Above, we have seen that $p \propto \rho^\gamma$, but the constant depends on the particle. Actually, we have

\[ p = \kappa e^{s/c_v} \rho^\gamma. \]

A fluid is called isentropic or barotropic, if there the specific entropy is a uniform constant, not just for a fluid path. As a result,

\[ p = p(\rho) = \kappa \rho^\gamma, \]

i.e. $p$ is a function of $\rho$.

Further, $\nabla p/\rho \propto \nabla h$ for isentropic fluid.

The Euler equation for isentropic fluids are then reduced to

\begin{align*}
  \rho_t + \nabla \cdot (\rho u) &= 0, \\
  (\rho u)_t + \nabla \cdot (\rho uu + \kappa \rho^\gamma I) &= b. \tag{0.3}
\end{align*}