KdV equation

We look at the KdV equations and the so-called integrable systems. The KdV equation can be written as

\[ u_t + \frac{3}{2} uu_x + \frac{1}{6} u_{xxx} = 0. \]

The constants 3/2 and 1/6 are not important as we can make them arbitrary by suitable scaling.

Boussinesq first obtained it in 1877. Korteweg and de Vries (1895) then derived it for weakly nonlinear shallow water waves.

In 1965, Zabusky and Kruskal derived the KdV equation from the FPU model

\[ m \frac{dx_n}{dt} = F(\Delta_0 + x_{n+1} - x_n) - F(\Delta_0 + x_n - x_{n-1}), \quad F(\Delta) = k\Delta + \alpha \Delta^3 \]

It arises from many models.

Derivation of KdV

Here, let us derive it from the Euler equations where the nonlinearity and dispersion reach a balance. It is a good model used to describe the tsunamis. The main reference is Section 3.2.1 in the book ‘A Modern introduction to the mathematical theory of water waves’.

Euler equations.

\[ u_t + u \partial_x u + w \partial_z u = -\partial_x (p/\rho), \]
\[ w_t + u \partial_x w + w \partial_z w = -\partial_z (p/\rho) - g \]

To be convenient, let us introduce

\[ P = \frac{p}{\rho} + gz - \frac{p_0}{\rho} \]

The equation becomes

\[ u_t + u \partial_x u + w \partial_z u = -\partial_x P, \]
\[ w_t + u \partial_x w + w \partial_z w = -\partial_z P \]

Suppose that the typical wavelength is \( \lambda \), the typical amplitude is \( a \) and the typical depth is \( h_0 \). The fluid region is

\[ 0 \leq z \leq h_0 + a\eta \]
The **shallowness** is $\delta = h_0/\lambda$, and the amplitude parameter is $\epsilon = a/h_0$. We will study the small amplitude, shallow regime. We focus on the particular regime $\delta^2 = \epsilon$.

We do the scalings
\[
\bar{x} = x/\lambda, \quad \bar{z} = z/h_0, \quad \bar{u} = u/\sqrt{gh_0}
\]

By the incompressibility, the scale for $w$ is $h_0\sqrt{gh_0}/\lambda$. The scale for pressure is $gh_0$.

Dropping the bars, we then find the scaled equations
\[
\begin{align*}
\epsilon (w_t + u \partial_x w + w \partial_z w) &= -\partial_z P, \\
\partial_\xi u + \partial_z w &= 0.
\end{align*}
\]

The boundary conditions
\[
P(z = 1 + \epsilon \eta) = \text{const} + \epsilon \eta, \quad w = \epsilon (\eta_t + u\eta_x)
\]

If one throws away the small terms, one gets the leading order terms
\[
\begin{align*}
u_t + u \partial_x u + w \partial_z u &= -\partial_x P, \\
0 &= -\partial_z P
\end{align*}
\]

and $z = 1$. The solution is trivial: $u = w = 0, P = \text{const}$.

Hence, to find the nontrivial solutions we must go to the next orders, especially for the dynamics of $\eta$. Further, the leading order should be a wave, so $\xi = x - t$ appears as a variable (using this variable, we are moving with the wave). We expect that the wave shape will change in a large time scale
\[
\tau = \epsilon t.
\]

These expectations allow us to do the multi-scalae ansatz
\[
q = \sum_{n=1}^{\infty} \epsilon^n q_n(\xi, \tau, z), \quad \eta = \sum_{n=0}^{\infty} \epsilon^n \eta_n
\]

since $u_0 = w_0 = 0, P_0 = \text{const}$. The equations are reduced to
\[
\begin{align*}
-\xi u + \epsilon \partial_\tau u + u \partial_\xi u + w \partial_z u &= -\partial_\xi P, \\
\epsilon (-\partial_\xi w + \epsilon \partial_\tau w) + \epsilon (u \partial_\xi w + w \partial_z w) &= -\partial_z P, \\
\partial_\xi u + \partial_z w &= 0
\end{align*}
\]

We also expand the boundary conditions
\[
P(\xi, \tau, 1 + (\epsilon \eta_0 + \epsilon^2 \eta_1 + \ldots)) = 1 + \epsilon \eta_0 + \epsilon \eta_1 + \ldots
\]
we find
\[ P_0(\xi, \tau, 1) = 1, \quad P_1(\xi, \tau, 1) = \eta_0(\xi, \tau, 1), \quad P_2(\xi, \tau, 1) + \eta_0 \partial_\xi P_1(\xi, \tau, 1) = \eta_1. \]

For the other boundary condition,
\[
(\epsilon w_1 + \epsilon^2 w_2)|_{z=1+\epsilon\eta_0 + \epsilon^2 \eta_1} + O(\epsilon^3) = \epsilon(-\eta_1 + \epsilon \partial_\tau \eta) + \epsilon^2 \eta_1 \partial_\xi \eta_0 + O(\epsilon^3)
\]
we find
\[ w_1(\xi, \tau, 1) = -\partial_\xi \eta_0, \quad w_2 + \eta_0 \partial_z w_1|_{z=1} = -\partial_\xi \eta_1 + \partial_\tau \eta_0 + u_1 \partial_\xi \eta_0 \]

- \(O(\epsilon)\),
  \[-\partial_\xi u_1 = -\partial_\xi P_1, \quad 0 = -\partial_z P_1, \quad \partial_\xi u_1 + \partial_z w_1 = 0.\]

The boundary condition for \(P_1(z = 1 + \epsilon \eta_1) = \eta_0\). Consequently,
\[ P_1 = \eta_0, \quad u_1 = \eta_0, \quad w_1 = -z \partial_\xi \eta_0 \]

The boundary condition \(w_1 = \partial_\tau \eta_0 = -\partial_\xi \eta_0\) is satisfied at \(z = 1\).

- \(O(\epsilon^2)\),
  \[-\partial_\xi u_2 + \partial_\tau u_1 + u_1 \partial_\xi u_1 + w_1 \partial_z u_1 = -\partial_\xi P_2, \]
  \[-\partial_\xi w_1 = -\partial_z P_2, \]
  \[\partial_\xi u_2 + \partial_z w_2 = 0\]

We find
\[ P_2 = \frac{1}{2}(1 - z^2) \partial_\xi \eta_0 + \eta_1. \]

and
\[ \partial_z w_2 = -\partial_\tau u_1 - u_1 \partial_\xi u_1 - w_1 \partial_z u_1 - \partial_\xi P_2 = -\partial_\tau \eta_0 - \eta_0 \partial_\xi \eta_0 - \frac{1}{2}(1 - z^2) \partial_\xi \eta_0 - \partial_\xi \eta_1 \]

The boundary condition
\[ w_2|_{z=1} = \eta_0 \partial_\xi \eta_0 - \partial_\xi \eta_1 + \partial_\tau \eta_0 + \eta_0 \partial_\xi \eta_0 \]
yields the KdV equation
\[ \partial_\tau \eta_0 + \frac{3}{2} \eta_0 \partial_\xi \eta_0 + \frac{1}{6} \eta_0 \partial_{\xi \xi \xi} = 0 \]

In other words, the change of wave profile in large time scale in the frame moving with the wave satisfies the KdV equation.
Mathematical structures of KdV

To be convenient, we do suitable scaling and study the KdV as

\[ u_t + uu_x + u_{xxx} = 0. \]

It is the Burger’s equation with a dispersive term. It is well-known that the nonlinear Burger’s equation can develop shocks. The dispersive term could balance the nonlinearity so that there is stable pump solution called a ‘soliton’. Solitons can go through each other without changing the shape.

This dispersive perturbation is very different from the dissipative perturbation

\[ u_t + uu_x = \epsilon u_{xx} \]

The latter is called, viscous Burger’s equation, which could be solved through the ‘Hopf-Cole’ transformation while the KdV can be solved through the ‘inverse scattering transformation’.

The dissipation (or equivalently white noise) could lead to the equipartition while the dispersive property may not lead to equipartition of energy.

**Remark 1.** For a linear operator with constant coefficients, \( A(u) \), one can consider the equation \( u_t = A(u) \). Since \( A(u) \) has eigenfunctions \( e^{ikx} \), then \( u_t = A(u) \) has solutions of the form \( e^{i(kx-\omega(k)t)} \). Any solution can be constructed as

\[ u = \int A(k)e^{i(kx-\omega(k)t)} dk. \]

\( \omega(k) \) is called the dispersion relation. If \( \omega(k) \) is real, then \( A(k)e^{-i\omega(k)t} \) does not decay and each mode travels with different speed. In this case, \( A(u) \) is called dispersive.

However, if \( \omega(k) \) is imaginary with negative imaginary part, then \( e^{-i\omega(k)t} \) decays. Such a term is called dissipative.

**Conservation laws and integrable systems**

Clearly, the KdV equation can be written as

\[ u_t + \left( \frac{u^2}{2} + u_{xx} \right)_x = 0 \]

We can also rewrite it as

\[ (u^2)_t + \left( \frac{2}{3}u^3 + 2uu_{xx} - u_x^2 \right)_x = 0 \]

This way, both \( \int u \) and \( \int u^2 dx \) are conserved.
For a given autonomous ODE system for $\vec{u} \in \mathbb{R}^N$:

$$\frac{d\vec{u}}{dt} = F(\vec{u}).$$

If we can find a conserved quantity, $E(u_1, \ldots, u_N) = C$, then we can reduce the ODE system to $N - 1$ dimension. Hence, if we can determine $N$ conserved quantities, the ODE system is totally solved by these $N$ constants. Of course, if the solution really depends on time, we may not be able to find such $N$ conserved quantities.

For a Hamiltonian, the story is quite different.

$$\frac{d}{dt} \vec{u} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \nabla_u H$$

where $\vec{u} \in \mathbb{R}^{2N}$. This is a $2N$-dimensional Hamiltonian system. The famous theorem of Liouville-Arnold is that if we can find $N$ conserved quantities (like energy, momentum etc) for a Hamiltonian, then, the solutions can be written in terms of these $N$ constants totally as quasiperiodic motion on the $N$-dimensional torus. Hence, if a $2N$-Hamiltonian has $N$ conserved quantities, it is called **completely integrable**. It is well-known that the three-body problem where one is with big mass is **not integrable**.

The KdV equation is also Hamiltonian, because it can be written as

$$u_t = J \frac{\delta H}{\delta u}, \quad J = \frac{d}{dx}, \quad H = - \int_{-\infty}^{\infty} \left[ -\frac{1}{6} u^3 + \frac{1}{2} u u_{xx} \right] dx.$$ 

($H$ is then automatically constant.) Well, is it completely integrable? The KdV equation is an infinitely dimensional Hamiltonian system.

Previously, we have seen that KdV has infinitely many conservation laws. A more clean way to see the infinitely many conservation laws is to adopt the **Gardner transform**.

**The Gardner transform:**

$$u = w + \epsilon w_x + A \epsilon^2 w^2.$$ 

Here, we consider the special case $A = -\frac{1}{6}$. Since $u$ satisfies the KdV, then $w$ satisfies the Gardner equation

$$w_t + w w_x - \frac{\epsilon^2}{6} w^2 w_x + w_{xxx} = 0 \Rightarrow w_t + \left( \frac{1}{2} w^2 - \frac{\epsilon^2}{18} w^3 + w_{xx} \right)_x = 0.$$ 

The idea is to solve $w$ in terms of $u$: we expand about $\epsilon$:

$$w = \sum_{n=0}^{\infty} \epsilon^n w_n[u]$$
where \( w_n[u] \) means it is a functional of \( u \). Comparing the coefficients of \( \epsilon \), we obtain:
\[
\sum_{n=0}^{\infty} \epsilon^n (\partial_t D_n + \partial_x F_n) = 0
\]
By this formula, \( \int D_n \, dx \) is a conserved quantity. Note that for odd \( n \), \( D_n \) is a total derivative so the odd terms are not interesting.

However, even though we have infinitely many local conservation laws, we may not still be able to conclude that KdV is completely integrable as we can not determine the solutions totally.

**Idea of Inverse scattering transform**

To solve the KdV totally, one can use the inverse scattering transform.

The Burgers’ equation \( u_t + uu_x = Au_{xx} \) can be solved by the Cole-Hopf: \( u = C\psi_x/\psi \) by choosing suitable \( C \) to reduce the Burgers to heat equation.

Gardner, Greene, Kruskal and Miura were trying to do the same thing for the KdV. They introduced
\[
 u + E = -6\psi_{xx}/\psi
\]
or
\[
 -6\psi_{xx} - u\psi = E\psi.
\]
This is the eigenvalue problem for Schrodinger equation of \( \psi \) with potential \( u \).

The key observation then is that if \( u \) solves the KdV, the eigenvalues \( E \) does not change with parameter \( t \).

If one starts solve the eigenvalue problem, there are two kinds of spectrum: the discrete spectrum and continuous spectrum.

If \( E = -\kappa_n^2, n = 1, 2, \ldots \), then the eigenfunction is in \( L^2(\psi) \) and asymptotically has the following form
\[
 \psi_n = c_n(t)e^{-\kappa_n x/\sqrt{\pi}}(1 + o(|x|)), \ x \to \infty.
\]
It is found that if KdV is satisfied \( c_n(t) = c_n(0)e^{A\kappa_n^3 t} \) for real \( A \).

If \( E = k^2 \) is positive, then the eigenfunctions are like planar waves:
\[
 \psi(x, k) = e^{ikx/\sqrt{\pi}} + r(k)e^{-ikx/\sqrt{\pi}} + o(1), \ x \to \infty,
\]
while
\[
 \psi(x; k) = t(k)e^{ikx/\sqrt{\pi}} + o(1).
\]
r is called the reflection coefficient while \( t \) is called the transmission coefficient. According to KdV, one can solve that
\[
 r = r(k; 0)e^{iNk^3 t}.
\]
The point is that if one knows the discrete spectrum $\kappa_n$, $c_n(t)$ and the reflection coefficient $r(k,t)$, one can recover the potential or the solution!
This is called the inverse scattering transform.

A brief word about Lax pair

Let
$$L = -6 \frac{d^2}{dx^2}, \quad B = -4 \frac{d^3}{dx^3} - u \frac{d}{dx} - \frac{1}{2} u_x.$$  

Then, Peter Lax found that the KdV equation can be rewritten as
$$\frac{dL}{dt} + [L, B] = 0.$$  

This is called the Lax equation.

In particular, if one considers the two linear equations
$$L\phi = \lambda \phi, \quad \phi_t = B\phi.$$  

The Lax equation is the consistent condition for these two linear equations. These two linear equations are called the Lax pair.

Using the Lax pair, one is able to construct the local conservation laws also and apply the inverse scattering transform to solve the solutions totally. Hence, if a Hamiltonian system has a Lax pair, then it is integrable.