Farfiled: multi-pole expansion

In studying the Stokes flows, we usually come up with integrals of the form

$$u_1 = \frac{1}{8\pi\mu} \int_S G \cdot f_S dS. \quad (1)$$

For example, $f_S = -\hat{n} \cdot \sigma$ is the force that $dS$ acting on the fluid where $\hat{n} = -\mathbf{n}$ is the normal vector pointing into the fluid.

Assume now that $|x| \gg 1$, what is the far field approximation?

To find this, we do Taylor expansion of $G(x - x')$ about the center $x$:

$$G(x - x') = G(x) - x' \cdot \frac{\partial G}{\partial x_l}|_{x' = 0} + \frac{1}{2} x'^l x'^m \frac{\partial^2 G}{\partial x_m \partial x_l}|_{x' = 0} + O\left(\frac{1}{|x|^4}\right)$$

We then find

$$u_1 = \frac{1}{8\pi\mu} G(x) \int_S f_S dS - \frac{1}{8\pi\mu} \frac{\partial G}{\partial x_l} \cdot \int_S x'^l f_S dS + \frac{1}{8\pi\mu} \frac{1}{2} \frac{\partial^2 G}{\partial x_m \partial x_l} \cdot \int_S x'^l x'^m f_S dS + O\left(\frac{1}{|x|^4}\right)$$

Let us look at the physical meanings of these terms.

$$F = \int_S f_S dS$$

is the total force acting on the fluid. Hence, the multiplying with the Green's function gives the Stokeslet contribution to the velocity field.

The second term is the gradient of Stokeslet, or force dipole. It is associated with the stresses and torques exerted on the fluid. The dipole strength tensor is

$$P = -\int_S x' f_S dS$$

As we did before, we can decompose $P$ as $P = D + \frac{1}{3} \text{tr}(P) I$ so that $D$ is traceless. Then, the force dipole can be decomposed as the stresslet and rotlet. For a neutrally buoyant particle, the total force and torque are zero and only the stresslet part survives.

The third term is the so-called quadrupole term, which we ignore.
2 Swimming problems of model microorganisms

2.1 Taylor’s swimming sheet

A simple model for the motion of flagellar: the swimming sheet. Imaging that there is a
wave propagating along the sheet and consequently the sheet obtains a swimming speed
$U \hat{x}$.

Now, we observe the sheet in the frame with velocity $U \hat{x}$. Then, the material coordinate
of a particle is given by

$$x_p = x, \ y_p = a \sin(kx - \omega t).$$

The boundary condition is that

$$\lim_{|y| \to \infty} u = -U \hat{x}.$$ 

**Remark 1.** We first note that this motion overcomes the scallop theorem. The reason is
that if you look back in time, you see a wave propagating in the opposite direction, so this
motion is not time-reversible.

To find $U$, we solve the Stokes equations in $y \geq a \sin(kx - \omega t)$. Introduce the stream-
function

$$u = \partial_y \psi, \ v = -\partial_x \psi.$$ 

By the no-slip boundary condition, we have

$$\partial_y \psi\big|_{y=a \sin(kx-\omega t)} = 0, \ \partial_x \psi\big|_{y=a \sin(kx-\omega t)} = a \omega \cos(kx - \omega t)$$

By the clear symmetry in time, we solve the equations at $t = 0$.

Scaling:

$$x' = kx, \ y' = ky, \ \psi' = \psi/(\omega a/k)$$

We assume small amplitude motion:

$$\epsilon = ka \ll 1.$$ 

The scaled problem becomes (dropping primes)

$$\Delta^2 \psi = 0$$

with boundary conditions

$$\partial_y \psi\big|_{y=\epsilon \sin(x)} = 0, \ \partial_x \psi\big|_{y=\epsilon \sin(x)} = \cos(x).$$
Now, we assume that $\psi$ has a regular perturbation in $\epsilon$:
\[
\psi = \sum_{n=0}^{\infty} \epsilon^n \psi_n = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \ldots
\]

Now, we deal with the boundary conditions:
\[
0 = \partial_y \psi|_{y=\epsilon \sin(x)} = \psi_y|_{y=0} + \epsilon \sin x \partial_{yy} \psi|_{y=0} + \ldots
\]
and
\[
\cos x = \partial_x \psi|_{y=\epsilon \sin(x)} = \partial_x \psi|_{y=0} + \epsilon \sin x \partial_{xy} \psi|_{y=0} + O(\epsilon^2)
\]
The leading order $O(1)$ is
\[
\Delta^2 \psi_0 = 0, \quad \partial_y \psi_0|_{y=0} = 0, \quad \partial_x \psi_0|_{y=0} = \cos x
\]
The solution can be found easily as we can use the form $\psi = f(y) \sin x$. The solution is
\[
\psi_0 = (1 + y)e^{-y} \sin x.
\]
For the next order $O(\epsilon)$, the boundary conditions read
\[
\partial_y \psi_1 + \sin x \partial_{yy} \psi_0 = 0, \quad 0 = \partial_x \psi_1 + \sin x \partial_{xy} \psi_0
\]
The equation is therefore
\[
\Delta^2 \psi_1 = 0, \quad \partial_y \psi_1 = (1 - y)e^{-y} \sin^2 x|_{y=0} = \sin^2 x, \quad \partial_x \psi_1 = ye^{-y} \sin x \cos x|_{y=0} = 0.
\]
The equation is linear and boundary condition is
\[
\partial_y \psi_1 = \frac{1}{2} - \frac{1}{2} \cos(2x)
\]
The solution is the superposition of these two modes. We thus find
\[
\psi_1 = \frac{1}{2} y - \frac{1}{2} ye^{-2y} \cos(2x)
\]
Hence,
\[
\psi = \psi_0 + \epsilon \psi_1 + O(\epsilon^2)
\]
Then, the velocity is
\[
-U' = \lim_{y \to \infty} \psi_y = \frac{1}{2} \epsilon + O(\epsilon^2).
\]
Recall,
\[
U' = U/[(\omega a/k)/(1/k)] = U/(\omega a) = U/(c \epsilon),
\]
we find
\[
U = -\frac{1}{2} \epsilon c^2 + O(\epsilon^3) \approx -\frac{1}{2} c k^2 a^2.
\]
Remark 2.  

- Note that \( U \) is not linear in \( a \) even though the equations are linear. The reason is that \( a \) appears in the boundary condition **not** in a linear fashion: on one side, it appears as \( a \omega \cos(kx - \omega t) \) and on the other hand, it appears in \( y = a \sin(kx - \omega t) \).

- Actually, we know \( U = U(a, \omega, k) \). If we change \( a \to -a \), we do not change the speed. The reason is that changing \( a \to -a \) corresponds a phase shift in the wave train and it does not change speed. Hence, we must know that \( U \) depends on the even powers of \( a \) only. A \( a^0 \) power is zero since if \( a = 0 \), there is no wave and the sheet does not swim. Hence the lowest order has to be \( e^2 \). This argument also applies to complex fluids.

- We have seen that the swimming speed is opposite to the wave speed \( c \) in the transverse wave case. However, if we consider a longitudinal wave, the situation is different. The no-slip condition is \( u(x + \delta(x, t), y = 0) = \frac{\partial}{\partial t} \delta = -a \omega \cos(kx - \omega t) \) but \( v = 0 \). The computed swimming speed is in the same direction as \( c = \omega/k \).

### 2.2 Local drag theory

The swimming sheet is for small amplitude motions. What if the amplitude is large? The question is as following: Suppose there is a line external distribution \( f_e \) on the flagellum, how does the flagellum move? (The external force could be the active force generated by the microorganism itself).

In this sense, we take advantage of the fact that the flagella are long and thin. Idea is to replacing the flagella with a line distribution of singular solutions.

The first theory using this idea is the **local drag theory** or resistive force theory. Let us look at non-rigorous but physically intuitive derivation.

Suppose the length of the flagellum is \( L \) and the typical radius is \( a \). We divide the flagellum into \( N \) segments and each segment is of radius \( a \). There are \( N = L/a \) segments.

Consider a ball at \( s_i \). We assume \( f_e \) and \( r(s) \) change in the scale \( L \). Then, on the interval \([s_i - c_1L, s_i + c_2L]\), \( f \) is kind of constant and these balls are aligned in a straight line with direction \( \hat{t} = \partial r/\partial s \). Outside this interval, the contribution is small, and we ignore.

The velocity field generated by itself is given by the Stokes law, \( \mathbf{u}_i = (f_i 2a)/(6\pi \mu a) \).

For other balls, the velocity field generated at the location \( i \) is \( \mathbf{u}_i^j = \frac{1}{8\pi \mu} \frac{1}{|x_i - x_j|} (1 + \hat{t} \hat{t}) \cdot f_j \Delta s \).
Hence, the total velocity is
\[ r_t = u_i = (f_{e,i}2a)/(6\pi\mu a) + \int_{-a}^{a} \frac{1}{8\pi\mu |s_i - s|} (1+i\hat{t}) \cdot f_e(s) ds + \int_{a}^{2a} \frac{1}{8\pi\mu |s_i - s|} (1+i\hat{t}) \cdot f_e(s) ds. \]

Then, by throwing away terms of order \( O(1) \) and smaller and only keeping the leading order term, we find
\[ r_t(s_i) \approx \ln(\frac{L}{a}) \frac{4\pi\mu}{(1 + i\hat{t}) \cdot f_e(s_i)} \]

This means to leading order, the velocity is determined by the local strength of force \( f_e \). The resistance coefficients are given by
\[ f_{e,\parallel} = -\xi \parallel u_\parallel, \quad f_{e,\perp} = -\xi \perp u_\perp \]
and
\[ \xi \perp = 2\xi \parallel = 4\pi\mu/\ln(L/a). \]

Later, Gray and Hancock derived more accurate resistance coefficients by placing Stokeslet and source dipoles:
\[ \xi \parallel = \frac{2\pi\mu}{\ln(2\lambda/a) - 1/2}. \]

Lighthill then proposed refined their arguments further.

**Remark 3.** The local drag theory is not accurate, with error \( O(1) \) but it can be used to derive swimming speeds approximately and give qualitative explanation. For more rigorous and accurate theory, one can refer to the slender body theory, as we shall introduce briefly next lecture.