Math 575-Lecture 19

In this lecture, we continue to investigate the solutions of the Stokes equations.

1 Energy balance

Rewrite the equation to

$$-\nabla \cdot \sigma = f.$$  

We begin the energy estimate by dotting $u$ in the Stokes equation and integrating:

$$\int_{\Omega} f \cdot udV = -\int_{\Omega} u \cdot (\nabla \cdot \sigma) dV = -\int_{S} u \cdot (n \cdot \sigma) dS + \int_{\Omega} \nabla u : \sigma dV$$  \hspace{1cm} (1.1)

where we have used the simple relation above. Note that $\sigma = -pI + 2\mu E$ where $E = \frac{1}{2}(\nabla u + \nabla u^T)$. Using $\nabla \cdot u = 0$, we reduce the relation to

$$\int_{S} u \cdot f_{S} dS + \int_{\Omega} f \cdot udV = 2\mu \int_{\Omega} E : EdV$$  \hspace{1cm} (1.2)

where $f_{S} = n \cdot \sigma = \sigma \cdot n$ is the force per unit area on the surface acting on the fluid. The obtained relation is nothing but the conservation of energy. The left hand side is the work done by the surface force and body force and the right hand side is the dissipation of energy. If the body force is conservative, the $f$ term is zero and the energy dissipation is given by the surface force only.

2 A variational formulation

We have seen that the energy dissipation is given by the following quantity

$$\Phi = 2\mu \int_{\Omega} E : EdV = \mu \int_{\Omega} (\nabla u + \nabla u^T) : \nabla udV.$$  

We now take the minimum of this functional subject to $\nabla \cdot u = 0$ and the boundary condition $u|_{S} = v_{S}$.

The formulation is to

$$L = \int_{\Omega} \mu(\nabla u + \nabla u^T) : \nabla u - \lambda \nabla \cdot udV$$

with a fixed boundary condition.
To take the variation, we require the variation $\delta u|_S = 0$ since the boundary conditions are imposed. We may take $\delta u = \epsilon w$. Then,

$$0 = \left. \frac{dL(u + \epsilon w)}{d\epsilon} \right|_{\epsilon = 0}$$

Consequently, we have

$$\int_{\Omega} \mu (\nabla w + \nabla w^T) : \nabla u + \mu (\nabla u + \nabla u^T) : \nabla wdV - \lambda \nabla \cdot wdV = 0$$

or

$$\int_{\Omega} 2\mu (\nabla u + \nabla u^T) : \nabla wdV - \lambda \nabla \cdot wdV = 0$$

Integrating by parts, we have

$$\int_{\Omega} -2\mu \Delta u \cdot w dV + \nabla \lambda \cdot wdV = 0$$

Here $w$ is no longer divergence free since we have used the Lagrange multiplier. Then the arbitrariness implies that

$$\nabla (\lambda/2) - \mu \Delta u = 0.$$ 

The derivative on $\lambda$ recovers the incompressibility condition.

We find that

$$p = \lambda/2$$

plays the role of Lagrange multiplier for the incompressibility condition.

3 Some particular situations

We now consider Stokes flows passing by cylinders or spheres.

3.1 2D: Stokes paradox

Stokes’ paradox:

Theorem 1. There is no non-trivial, steady state solution to Stokes’ equations in $\mathbb{R}^2$ in the region exterior to a disc.

To illustrate, let us consider the uniform flow passing the cylinder with radius $a$. Taking the curl on the Stokes equations, we get

$$\Delta \omega = 0$$
Since $\omega = -\Delta \psi$ where $\psi$ is the 2D streamfunction, we have
$$\Delta^2 \psi = 0.$$ 

The streamfunction satisfies the biharmonic equation. 

As $r \to \infty$, $\psi \to U r \cos \theta$. The equation is linear in $\theta$ with constant coefficient, so there is no new modes generated, and we can just use the ansatz
$$\psi = f(r) \sin \theta.$$ 

The Laplacian is
$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$ 

Consequently, we have
$$f(r) = Ar^3 + Br \ln r + Cr + Dr^{-1}.$$ 

Matching the boundary conditions, $A = B = 0$. The no-slip condition requires that $C = D = 0$. (Note $u = \nabla \times (\psi \hat{z}) = (\hat{r} \partial_r + \hat{\theta} \partial_\theta + \hat{z} \partial_z) \times (\psi \hat{z})$)

For the reason, see the words on P117-P118 in Childress. Roughly speaking, the reason is that in 2D outside the cylinder, $Re u \cdot \nabla u$ is comparable to $\Delta u$ no matter how small $Re$ is. Outside a distance $aRe^{-1}$, we should use the Navier-Stokes equations or Oseen’s equations to approximate. In 3D, there no such an issue.

### 3.2 3D: Uniform Stokes flow past a sphere

We use the spherical coordinates $(\rho, \theta, \phi)$ such that
$$x = \rho \sin \theta \cos \phi, \ y = \rho \sin \theta \sin \phi, \ z = \rho \cos \theta.$$ 

Introduce the Stokes stream function in Lecture 10 $\Psi(\rho, \theta)$, we can rewrite the velocity field as
$$u = \frac{\partial \Psi}{\rho^2 \sin \theta} \hat{\rho} - \frac{\partial \Psi}{\rho \sin \theta} \hat{\theta}.$$ 

By the Stokes equation: we find
$$\mu \Delta \omega = 0.$$ 

or
$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial \omega}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \omega}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 \omega}{\partial \phi^2} = 0.$$ 

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As in Lecture 10, we find that

$$\omega = -\hat{\phi} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial \rho \Psi}{\rho \sin \theta} \right) + \frac{\partial \rho \Psi}{\rho^2 \sin \theta} + \frac{1}{\rho} \frac{\partial \theta}{\partial \theta} \left( \frac{\partial \theta \Psi}{\rho^2 \sin \theta} \right) \right) = \omega \hat{\phi}. $$

Consequently, we have

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \frac{\partial \omega}{\partial \rho}) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \omega}{\partial \theta}) - \frac{\omega}{\rho^2 \sin^2 \theta} = 0. $$

Recall the boundary conditions $u \to U \hat{z}$ as $|x| \to \infty$. In cylindrical,

$$\psi(r, z) \to \frac{1}{2} U r^2, \sqrt{r^2 + z^2} \to \infty$$

or

$$\Psi \to \frac{1}{2} U R^2 \sin^2 \theta, \ R \to \infty. $$

For completeness, one could try the separation of variables. Here, we may simply try the form

$$\Psi = \sin^2 \theta f(\rho) $$

(Note that there is $\sin \theta$ in the equation so there could be new modes generated.)

Consequently,

$$\omega = -((f'/\rho)' \sin \theta + \sin \theta \frac{1}{\rho^2} f' - \frac{f'}{\rho^3} (2 \sin \theta)) = -\sin \theta (f''/\rho - 2 f/\rho^3) = -\sin \theta h(\rho). $$

Inserting this into the equation, we find

$$\frac{1}{\rho^2} (\rho^2 h')' \sin \theta + h \frac{1}{\rho^2 \sin \theta} (\sin \theta \cos \theta)' - \frac{h}{\rho^2 \sin \theta} = 0 $$

or

$$\frac{1}{\rho^2} (\rho^2 h')' - \frac{2h}{\rho^2} = 0. $$

This means that the assumption that $\Psi$ has a single mode is reasonable.

The equation for $h$ is simplified as

$$\rho^2 h'' + 2 \rho h' - 2h = 0 $$

This is an Euler ODE and the general solution is of the form $\rho^n$. We find $n = -2, 1$. Hence,

$$h(\rho) = A \rho + B \rho^{-2} $$
Hence,
\[ \rho^2 f'' - 2f = \rho^3 h = A\rho^4 + B\rho \]
This is again Euler ODE with forcing term. The general form for \( f \) is thus given by
\[ f(\rho) = C_1 \rho^{-1} + C_2 \rho + C_3 \rho^2 + C_4 \rho^4. \]

Comparing with the limit as \( \rho \to \infty \), we must have \( C_4 = 0 \) and \( C_3 = \frac{1}{2}U \). To determine \( C_1, C_2 \), we use the boundary conditions at \( \rho = a \).

We have \( \partial_\theta \Psi = 0, \partial_\rho \Psi = 0 \) at \( \rho = a \). Hence, \( f(a) = f'(a) = 0 \). Consequently, we have
\[ C_1 = \frac{1}{4} Ua^3, \quad C_2 = -\frac{3}{4} Ua. \]

The Stokes streamfunction is
\[ \Psi = \frac{1}{4} U(a^3 \rho^{-1} - 3a\rho + 2\rho^2) \sin^2 \theta. \]