Math 575-Lecture 18

Now, we move to another world: vanishing \( Re \) regime: \( Re \sim 0 \). Recall that

\[
Re = \frac{\rho UL}{\mu}.
\]

This can be achieved with

- very viscous fluid
- very small length scales,
- creeping flows (with small velocity).

Recall that the scaled equations read

\[
Re \left( \partial_t u + u \cdot \nabla u \right) = -\nabla p - \Delta u + f.
\]

**Remark 1.** Note that the pressure and body force terms are scaled differently compared with the one in \( \partial_t u + u \cdot \nabla u = -\nabla p - \frac{1}{Re} \Delta u + f \). Which scales for pressure and force should be used clearly depends on the regimes we consider. In the low \( Re \) world, \( \nabla p \) and \( f \) are usually big and \( Re \nabla p \) and \( Re f \) are of order \( O(1) \). Hence, the appropriate scalings in \( Re \) regimes should be the one we consider.

If we consider \( Re \ll 1 \), then the inertial terms are unimportant compared with the viscous term. Hence, for these regimes, we can approximate the dynamics of the fluids by solving

\[
\nabla p - \mu \Delta u = f, \quad \nabla \cdot u = 0.
\]

The system of equations is referred to as ‘Stokes equations’. The special dynamical regime is referred to as ‘Stokes flows’.

The boundary conditions are the no-slip boundary conditions as before for viscous fluid. Specifically, if the solid surface is not moving, then we have

\[
u|_{\partial \Omega} = 0.
\]

We now list some properties implied by the Stokes equations:

**Linearity**

Let \( f = 0 \). If \((u_1, p_1)\) and \((u_2, p_2)\) are both solutions, then so is

\[
(\alpha u_1 + \beta u_2, \alpha p_1 + \beta p_2),
\]

where \( \alpha \) and \( \beta \) are arbitrary constant scalars.

Note that if \( f \) is conservative, then it can be absorbed into pressure and we can think \( f = 0 \) in this case.
Uniqueness of solution

Consider a domain $\Omega$ with boundary $S = \partial \Omega$. Given the boundary conditions $v_S$. If the domain is unbounded, we require that there is a velocity field $u_\infty(x)$ such that $u - u_\infty \to 0$ and $\nabla(u - u_\infty) \to 0$ fast enough. $u_\infty$ can be regarded as the boundary condition at $\infty$.

Claim: The solution is unique.

Suppose there are two solutions $(u_1, p_1), (u_2, p_2)$, with the same boundary conditions. Then, $v = u_1 - u_2$ also solves the Stokes equations with pressure $p = p_1 - p_2$ but with zero boundary conditions.

Hence, we find

$$0 = \int_\Omega v \cdot (\nabla p - \mu \Delta v) dV = \mu \int_V |\nabla v|^2 dV.$$

Then, $v = 0$ by the zero boundary condition.

Instantaneous response

Diffusion of velocity perturbation is instantaneous. Fluid responds to the motion of the boundaries immediately. This is because the equations do not contain time explicitly. For a given boundary condition, we can solve the solutions at that time.

Force balance and torque balance

Since the inertial terms are neglected, the total force and total torque acting on a body immersed in a Stokes flow must be zero.

Mathematically, consider the equation

$$\nabla \cdot \sigma + f = 0.$$

Integrating in $\Omega$, we find

$$\int_S n \cdot \sigma + \int_V f dV = 0$$

or

$$\int_S t dS + \int_V b f dV = 0.$$

If there is no external force, we must have

$$\int_S t dS = 0.$$
The torque is defined as

\[ L = \int_S \mathbf{x} \times \mathbf{t} dS \]

while the external torque is

\[ L_e = \int_V \mathbf{x} \times \mathbf{f} dV. \]

Then, we compute

\[
\int_\Omega \mathbf{x} \times \nabla \cdot \sigma dV = \int_\Omega \epsilon_{ijk} x_i \partial_m \sigma_{mj} dV = \int_\Omega \epsilon_{ijk} \partial_m (x_i \sigma_{mj}) - \epsilon_{ijk} \delta_{im} \sigma_{mj} dV = \int_\Omega \epsilon_{ijk} \partial_m (x_i \sigma_{mj}).
\]

Then, applying the divergence theorem, we find that

\[ L = \int_\Omega \mathbf{x} \times \nabla \cdot \sigma dV. \]

Reversibility

Reverse the boundary conditions

If \( \mathbf{v} \) is the flow driven by \( \mathbf{f} \) and boundary condition \( \mathbf{v}_S \), then \(-\mathbf{v}\) is the flow driven by \(-\mathbf{f}\) and \(-\mathbf{v}_S\) by the uniqueness of Stokes flows.

Let us consider examples:

- Consider an object in the Stokes flow which tends to a uniform flow at infinity. The object feels a drag \( \mathbf{F} \). If we rotate the object clockwise with angle \( \pi \), what is the drag that the body feels?

  Consider that we do not rotate the object. Instead, we reverse the velocity at infinity. Then, the boundary conditions \( \mathbf{v}_S \) are reversed since the velocity on the surface of the body is zero. Then, the whole velocity field and pressure gradient change signs. The drag is reversed. However, if we view the whole object in another direction, we see the problem we consider. Hence, the drag is the same.

- Consider two identical spheres sedimenting in a Stokes flow in the same vertical line. Are their velocities the same?

  If we have more than two spheres, the middle ones move faster.

If we have more than two spheres, the middle ones move faster.
**Time-reversal symmetric periodic boundary motions**

The velocity field in Stokes flow is determined by the boundary condition totally.

We first make a useful observation: suppose there are two kinds of boundary motions

\[ S(t) : 0 \leq t \leq T \text{ and } S_2(t) : 0 \leq t \leq T, \text{ and } S_2(t) = S(\tau(t)) \]

for some differentiable continuous function \( \tau, \tau(T) = T \).

We claim that all the fluid particles under \( S(t) \) at time \( t \) are at the same positions as the fluid particles under \( S_2(t) \) at \( \tau^{-1}(t) \).

The is clear since the boundary condition under \( S_2 \) is given by \( v_S(\tau) \dot{\tau} \). Hence, the solution is simply \( u \dot{\tau} \).

\[
D_2 = \int_0^{\tau^{-1}(t)} u \dot{\tau} \, ds = \int_0^t u \, d\tau = D_1.
\]

This means on the sequence of boundary configuration matters for the displacement. The time of the sequence is not important.

Consider a periodic boundary motion that exhibits time-reversal symmetry: i.e. if we look back in time, we see the same sequence of shapes. Then, we have the following

**Theorem 1.** In Stokes flow, time-reversal symmetry of periodic boundary motion implies that all fluid particles return periodically to their starting positions.

One of the applications of Stokes equations is to study the swimming of micro-organisms. We have the following so-called ‘Scallop theorem’

**Theorem 2.** if the sequence of shapes displayed by a swimmer deforming in a time-periodic fashion is identical when viewed after a time-reversal transformation, then the swimmer cannot move on average

The idea is essentially the same. For the detailed explanation, read ‘The hydrodynamics of swimming microorganisms’. The Scallop theorem puts severe constraint on the strategies to move for microorganisms. For the usual scallop, it moves like... However, the same strategy can not make it move in the Stokes flows.

In a word

- Based on drag instead of by momentum transfer.
- Must overcome the Scallop theorem(either the motion is irreversible or the fluid is complex)