Math 575-Lecture 17

We continue the discussion for boundary layers. In particular, we will find solutions to the Prandtl boundary layer equations for some particular cases.

Recall that the boundary layer equations read

\[ \partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \nu \partial_y^2 u, \]
\[ \partial_y p = 0, \]
\[ \partial_x u + \partial_y v = 0, \]
\[ u = 0 = v, y = 0. \]

\( \nu \) is small but not zero.

In this section, we study stationary problems and we adopt the scaled version of equations (Recall that \( H/L = Re^{-1/2} \) for boundary layer):

\[ x' = x/L, \ y' = y\sqrt{Re/L}, \ u' = u/U, \ v' = v\sqrt{Re/U}. \]

To be convenient, we ignore the prime notations from here on in this section.

\[ u \partial_x u + v \partial_y u = -\partial_x p + \partial_y^2 u, \]
\[ \partial_y p = 0, \]
\[ \partial_x u + \partial_y v = 0, \]
\[ u = 0 = v, y = 0. \]

1 Blasius solution for a semi-infinite flat plate

We now consider a semi-infinite plate which is represented by the positive \( x \)-axis. Consider that the flow outside the boundary layer is \( u = \langle U, 0 \rangle \) so that \( p = p_0 \) in the outer solution.

We now solve the solution inside the boundary layer. For the boundary layer, as \( y \to \infty \), it should match the outer solution, or \( u \to U \) as \( y \to \infty \). After scaling \( u \to 1 \) as \( y \to \infty \).

According to the second equation, \( p = p_0 \) inside the layer as well.

Consequently,

\[ u \partial_x u + v \partial_y u = \partial_y^2 u \]

By the incompressibility condition, we are able to find a streamfunction

\[ \psi(x, y) \]

such that \( \langle u, v \rangle = \langle \psi_y, -\psi_x \rangle \). Note that \( Uu' = u \Rightarrow U \partial \psi / \partial y' = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y} \frac{Re^{1/2}}{L} \). We find that the dimensional streamfunction is given by

\[ \Psi = ULRe^{-1/2} \psi \]
Hence, we have the equation reduced to
\[ \psi_y \psi_{xy} - \psi_x \psi_{yy} - \psi_{yyy} = 0. \]

To solve this equation, we first investigate the scaling problems and then find the self-similar solutions.

If we do the scaling \( \bar{x} = \lambda x, \bar{y} = \lambda^\alpha y \) and \( \bar{\psi} = \lambda^\delta \psi \), we find that the equation is scaling-invariant if \( 2\alpha + 1 - 2\delta = -\delta + 3\alpha \) or \( \alpha + \delta = 1 \).

Further, \( u = \psi_y \rightarrow 1 \) as \( y \rightarrow \infty \). We do not rescale this boundary condition, and then we require \( \lambda^{\delta - \alpha} = 1 \). Hence, we find \( \alpha = \delta = 1/2 \). In other words, the scaling is as \( \psi(x,y) \rightarrow \lambda^{1/2} \psi(\lambda^{-1/2}x, \lambda^{-1/2}y) \)

If we drop bars, we conclude that if \( \psi(x,y) \) is a solution, then \( \lambda^{1/2} \psi(\lambda^{-1}x, \lambda^{-1/2}y) \) is also a solution. This means one may find the self-similar solution of the form
\[ \psi(x,y) \rightarrow \lambda^{1/2} \psi(x, y) \]

Inserting this ansatz into the equation, we then have
\[ F'(F'' \left( -\frac{y}{2x^{1/2}} \right)) - \left( \frac{1}{2x^{1/2}} F(\eta) - \frac{y}{2x} F'(\eta) \right) F''(\eta) \left( \frac{1}{x^{1/2}} - \frac{1}{x} F'''(\eta) = 0 \right. \]

or
\[ \frac{1}{2} F'' + F''' = 0. \]

For the boundary conditions, we find \( F(0) = 0 \) and \( F'(0) = 0 \) since \( u = 0 \) on \( y = 0 \). Finally, \( u = \psi_y \rightarrow 1 \) as \( y \rightarrow \infty \). Consequently, \( \lim_{\eta \to \infty} F'(\eta) = 1 \).

The solution of this ODE can be found easily.

If we scale back, we have the dimensional stream function
\[ \Psi = ULR e^{-1/2}(x/L)^{1/2} F\left( \frac{Re^{1/2}y/L}{\sqrt{x/L}} \right) = \sqrt{U \nu x} F(y \sqrt{U \nu x}) \]

As \( y \rightarrow \infty \),
\[ \Psi = U y - 1.7208 \sqrt{U \nu x} + o(1). \]

This means asymptotically \( y = 1.7208 \sqrt{\nu x/U} \) is a streamline. The plate then seems to have a thickness, which grows like \( \sqrt{x} \). This thickness is called the ‘displacement thickness’.

(For more discussions, see Sec. 8.2.1 in Childress.)
2 Falkner-Skan family of boundary layers

Consider that the outer solution is not a uniform flow. Instead, consider the flow down a wedge, with angle $\alpha = \frac{m}{m+1}\pi$, and then transits to horizontal plate for $x > 0$.

Recall that the mapping

$$Z(z) = z^{\pi/(\pi-\alpha)} = z^{m+1}$$

maps the region above this solid boundary into the upper half plane. The complex potential for the ideal fluid in the $Z$ plane with velocity $U$ at $|Z| = \infty$ is given by $F(Z) = UZ$. Hence,

$$f(z) = Uz^{m+1}$$

yields a complex potential for the ideal fluid in the wedge region. We compute that

$$u - iv = (m + 1)Uz^m$$

Hence, the velocity on $x$-axis is given by $(u, v) = (Ax^m, 0)$ where $A$ is a constant. As $x \to \infty$, the velocity does not converge to a finite velocity. Instead, it tends to infinity.

Using the Bernoulli principle, we find that

$$p_x = -mA^2x^{2m-1}$$

We now use this solution as the outer solution in the boundary layer problem and figure out the inner solution inside the boundary layer.

The first equation in the Prandtle is given by

$$uu_x + vu_y - mA^2x^{2m-1} - \nu u_{yy} = 0$$

As before, we use the streamfunction $\psi$ and do the self-similar solution

$$\psi = x^\alpha G(y/x^\beta)$$

Our goal is to insert this self-similar solution and then get an ODE.

The boundary is a streamline and thus $G(0) = 0$. Also, $u = 0$ on $y = 0$ so $G'(0) = 0$. Further, it is expected that $\lim_{y \to \infty} \psi_y = Ax^m$. Hence, we need $\alpha - \beta = m$.

If we insert, we find that the equation is scaling invariant if

$$\alpha = \frac{1 + m}{2}, \quad \beta = \frac{1 - m}{2}.$$ 

If one introduces $K = \sqrt{\frac{\nu}{(m+1)A}}$ and $\eta = y/(Kx^{(1-m)/2})$, and $\psi = AKx^{(1+m/2)}F(\eta)$,
then
\[ F''' + \frac{1}{2} FF'' + \frac{m}{1+m}(1-F'^2) = 0 \]

Then, \( F(0) = 0 = F'(0) \) and \( \lim_{y \to \infty} F'(\eta) = 1 \).

In the case \( m > 0 \), \( U'(x) > 0 \) and \( p'(x) < 0 \). Favorable pressure gradient. (The boundary layer is stable and there is no separation of boundary layer.)

If \( m < 0 \), uniqueness of the profile is not true (there may be several solutions). Note that this does not contradict with the existence and uniqueness theorem from ODE theory since it is not an initial value problem. However, if one requires \( u \geq 0 \) for \( x > 0 \), the profile is unique. \( F'' = 0 \) implies that \( u_y = 0 \) or \( \omega = 0 \). The pressure gradient could be positive near this point. Then, there could be separation of boundary layer. This is then called the unfavorable case.

### 3 A 2D laminar jet

Consider a laminar 2D steady jet from a small slit in a wall. If \( Re \gg 1 \), the jet is keeps to be thin. (In experiment, if \( Re \) is big enough, the jet becomes turbulent. Here we do not consider the turbulent jet.)

In the model, the slit is considered as a point. Since the jet is thin, we can apply the Prandtl equations.

The solution outside the jet is considered as stationary: \( u = 0 \) and \( p = p_0 \). By the second equation in Prandtl, \( p \) is a constant inside the jet. Hence, we have Consequently,

\[ u \partial_x u + v \partial_y u = \nu \partial_y^2 u \]

By the incompressibility condition, we are able to find a streamfunction

\[ \psi(x, y) \]

such that \( \langle u, v \rangle = \langle \psi_y, -\psi_x \rangle \). Hence, we have the equation reduced to

\[ \psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0. \]

Again the self-similar solution is of the form (this choice picks \( \psi = 0 \) on \( y = 0 \)):

\[ \psi = x^\alpha F(y/x^\beta) \]

Like in the Blasius case, we need \( \alpha + \beta = 1 \) for the solution to be self-invariant. However, the case that is different here is that we do not need \( \lim_{y \to \infty} u = U \). The limit here is zero. This zero limit can not put a constraint on \( \alpha \) and \( \beta \).
The constraint is that $p$ is a constant. The momentum flux
\[ M = \int_{-\infty}^{\infty} u^2 dy \]
must be independent of $x$ since the forces acting on both sides of a vertical strip $[x, x + dx] \times [-\infty, \infty]$ cancel.

This constraint requires that
\[ \beta = 2\alpha. \]

Hence,
\[ \psi = x^{1/3} F(\eta), \eta = y / x^{2/3}. \]
Consequently,
\[ \nu F^\prime\prime\prime + \frac{1}{3} (FF')' = 0. \]

For the boundary conditions, we only have $F'(\eta) \to 0$ as $\eta \to \infty$. Further, we also ask the vorticity to vanish: $F''(\eta) \to 0$ as $\eta \to \infty$.

Integrating once,
\[ \nu F^\prime\prime + \frac{1}{3} FF' = 0 \]
Integrating again:
\[ \nu F' + \frac{1}{6} F^2 = \frac{1}{6} F(\infty)^2. \]

Hence with $F(\infty) = F_\infty$,
\[ F(\eta) = F_\infty \tanh(\frac{F_\infty \eta}{6\nu}) \]

To determine $F_\infty$, we must use the fact
\[ M = \int u^2 dy = \int (F')^2 d\eta = M \]
or
\[ M = \frac{2F_\infty^2}{9\nu} \]

As long as we find $\psi$, the velocity is given as
\[ u = \frac{F_\infty^2}{6\nu x^{1/3} \cosh^2(F_\infty \eta / (6\nu))}. \]

The velocity decays as $x^{-1/3}$.

4 **Prandtl-Batchelor theory**

The boundary layer theory is for large $Re$ theory. See the Prandtl-Batchelor theory for $Re \to \infty$ limit of solutions on P134 in Childress.