Math 575-Lecture 16

Now, we consider the N-S equations:

\[ \partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \]
\[ \nabla \cdot u = 0, \]
\[ u = 0, \ x \in \partial \Omega. \]

where \( \nu = \mu/\rho \).

If the viscosity \( \nu \to 0 \), then \( Re \to \infty \). Formally, \( \mu \to 0 \) yields Euler equations, but, as we have seen, the limit of vanishing viscosity is not exactly the same as Euler equations, since the viscosity can lead to vortex shedding even if in the \( \mu \to 0 \) limit. This is mainly due to a boundary layer near the solid surface. (Away from the vortical wakes and the solid surface, the \( \mu \to 0 \) limit yields the Euler equations but these thin layers and wakes are important.)

In the following two lectures, we aim to study in detail how different a flow with large \( Re \) is from the one described by Euler equations, and how the boundary layer behaves.

1 An example of boundary layer in ODE

Consider the ODE with ‘viscosity’:

\[ \nu y''(x) + y'(x) = a, \ y(1) = 1, \ y(0) = 0. \]

If we take formally \( \nu \to 0 \), we obtain the ‘Euler equations’:

\[ y'(x) = a, \ y(1) = 1. \]

1.1 The accurate solution

For the second order ODE, we can solve accurately that

\[ y(x) = \frac{1 - a}{1 - e^{-1/\nu}} (1 - e^{-x/\nu}) + ax. \quad (1.1) \]

The solution to the first order ODE is given by

\[ y(x) = 1 - a + ax. \quad (1.2) \]

If we plot the graphs, we find that the solution of the second order ODE is close to the solution to the first order ODE in most regions but there is a thin layer within which the solution changes rapidly. This is because the solution wants to satisfy the boundary condition but \( 1 - a + ax \) does not give the correct boundary condition at \( x = 0 \).
1.2 An asymptotic analysis and matching

We now perform the boundary layer analysis.

The idea is as follows: Step 1: Have an idea of the location of the boundary layer. Step 2. Solve the inner problem (the problem inside the layer). In this step, we determine the width scale of the boundary layer, and solve the leading order equation. Step 3. Solve the outer problem with the corresponding boundary condition. Usually, in the outer problem, all derivatives are of order $O(1)$. 4. Match the inner solution and outer solution, and find a uniformly accurate solution.

- The boundary layer is near $y = 0$. Assume the boundary layer is of length $\delta$ and we introduce the inner variable $z = x/\delta$. Then,

$$\nu \frac{1}{\delta^2} y_{zz} + \frac{1}{\delta} y_z = a$$

$y_{zz}, y_z$ are $O(1)$ if we assume that the boundary layer is resolved well. There are two dominate terms that should balance. If $\frac{1}{\delta} y_z \sim a$, then $\delta = 1$, which is not the boundary layer. In the sense that $\delta \ll 1$, $\frac{1}{\delta} y_z$ be bigger than $a$. Hence, we must have $\nu \frac{1}{\delta^2} \sim \frac{1}{\delta}$ or $\delta = \nu$.

Hence, the leading equation is

$$y_{zz} + y_z = 0, \quad y|_{z=0} = y(0) = 0, \quad y|_{z=\infty} = y_{out}(x \to 0+)$$

We find $y(z) = C_1 e^{-z} + C_2$ with $C_1 + C_2 = 0$.

- In the outer solution, $y_{xx}, y_x$ are all $O(1)$, and the leading order equation is given by

$$y'(x) = a, \quad y(x) = ax + b$$

With the boundary condition, $b = 1 - a$. Hence, the outer solution is $y_{out}(x) = ax + 1 - a$. If $x \to 0$, we find $y_{out}(0) = 1 - a$.

- We then have

$$C_2 = 1 - a, \quad C_1 = a - 1.$$ 

Hence, the inner solution is

$$y_{in}(x) = (a - 1)e^{-x/\nu} + (1 - a)$$

The uniform solution is

$$y = y_{in} + y_{out} - (1 - a) = (a - 1)e^{-x/\nu} + ax + 1 - a$$

where we substract $1 - a$ because it is the common limit and we only want to count it once. In the region $x > \nu$, $y_{in} - (1 - a)$ is nearly zero and $y_{out}$ is the main effect. In the region $x < \nu$, $y_{out} - (1 - a)$ is nearly zero and $y_{in}$ is the main effect.
2 High $Re$ and Prandtl boundary layer equations

As we have seen in the Rayleigh’s problem or the stagnation point flow problem, there is a transition layer of thickness $\sim \sqrt{1/Re}$ near the solid boundary.

We assume there is a thin layer of thickness $H \ll L$ ($H$ plays the role of $\delta$ as above) so that the fluid transits into the nearly ideal fluid if $Re \gg 1$ or $\nu \ll 1$. We use $H$ just to compare with the lubrication theory.

To be convenient, we consider a 2D incompressible homogeneous fluid in the $y > 0$ plane. Suppose that we rescale the quantities as follows

\[ x' = x/L, \quad y' = y/H, \quad t' = t/T, \quad u' = u/U, \quad v' = v/(HU/L), \quad p' = p/P. \]

The incompressibility condition is reduced to

\[ \partial_{x'} u' + \partial_{y'} v' = 0. \]

Both terms are equally important.

\[ \frac{U}{T} \partial_{x'} u' + \frac{U^2}{L} (u' \partial_{x'} u' + v' \partial_{y'} u') = -\frac{P}{L} \partial_{x'} p' + \frac{\nu U}{H^2} \left( \frac{H^2 \partial^2 u'}{L^2 \partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} \right) \]

\[ \frac{UH}{TL} \partial_{x'} v' + \frac{U^2 H}{L^2} (u' \partial_{x'} v' + v' \partial_{y'} v') = -\frac{P}{H} \partial_{y'} p' + \frac{\nu U}{HL} \left( \frac{H^2 \partial^2 v'}{L^2 \partial x'^2} + \frac{\partial^2 v'}{\partial y'^2} \right) \]

Unlike the lubrication, here, we are requiring that the viscous term is balancing the inertial term.

\[ \frac{\nu U}{H^2} \sim \frac{U^2}{L} \]

or

\[ O(1) = Re = \left( \frac{H}{L} \right)^2 \frac{UL}{\nu} = \left( \frac{H}{L} \right)^2 Re \]

Hence,

\[ H/L \sim O(Re^{-1/2}). \]

Consequently, $P = \nu UL/H^2$. Hence, keeping the important terms, we have

\[ \partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \nu \partial_y^2 u, \]

\[ \partial_y p = 0, \]

\[ \partial_x u + \partial_y v = 0, \]

\[ u = 0 = v, \quad y = 0. \]

(P72-P76 on C.M.)

This system of equations is called the **Prandtl boundary layer equations**.

The solution is therefore the **inner solution** and finally one wants to match the outer solution, which is given by the Euler equations.
3 Vorticity and the boundary layer

Clearly, \( p = p(x,t) \) by the second equation. Hence, the pressure should be equal to the boundary value of pressure for the outer solution.

Now, we investigate the fact that the boundary layer can produce vorticity.

We first consider the rectangle \( ABCD \). \( AB \) is parallel to \( x \)-axis at \( y = y_0 \) and \( CD \) is on \( x \)-axis. \( BC, AD \) are perpendicular to the solid surface. The circulation is given by

\[
\Gamma = -\left( \int_{AB} u \, dx + \int_{BC} v \, dy + \int_{CD} u \, dx + \int_{DA} v \, dy \right)
\]

since \( v \) is small in the layer, and \( u = 0 \) on \( y = 0 \), we find

\[
\Gamma \approx -\int_{AB} u \, dx < 0.
\]

Hence, there is negative vorticity in the boundary layer. The layer actually generates vorticity.

To confirm this more, let us note

\[
\omega = \partial_x v - \partial_y u \sim -\partial_y u.
\]

Taking \( y \) derivative in the first equation for Prandtl boundary layer equation, we have

\[
\partial_t \omega = \frac{1}{R} \partial_y^2 \omega - u \partial_x \omega - v \partial_y \omega
\]

Here, though \( v \) is small, \( \partial_y \) is large. Hence, \( u \partial_x \omega \) and \( v \partial_y \omega \) are of the same importance. This equation tells us that the vorticity on one side is convected downstream and at the same time diffused vertically into the fluid from the boundary layer.

Now, let us consider a special case so that we can find the the solution to the Prandtl boundary layer equations.

Suppose the velocity at \( y = \infty \) is given by \( U \hat{x} \). Then, we can assume that the velocity is of the form

\[
u = \langle u(y,t), 0 \rangle.
\]

Further, if there is no pressure gradient \( p = p(y,t) \), then we find

\[
\partial_t u = \nu \partial_y^2 u
\]

For this equation, we can pursue the self-similar solution of the form

\[
u(y,t) = U f(y/\sqrt{t})
\]

subject to the boundary condition \( f(0) = 0, \lim_{\xi \to \infty} f(\xi) = 1 \).
We find

\[ u(y, t) = 2U \frac{1}{\sqrt{\pi}} \int_0^{y/(2\sqrt{\nu t})} e^{-s^2} ds. \]

The vorticity is \( \omega = -\partial_y u \). For a fixed \( t \), the plot is roughly as ...

We find that the vorticity is everywhere negative and decays to almost zero outside the boundary layer.

### 4 Separation of boundary layers

It is observed that the boundary layer could separate from the solid body if the surface is curved. One crucial part in the classical fluid dynamics is to understand the mechanisms of the separation of boundary layer.

Heuristically, the boundary layer separates at the point where

\[ \omega = -\partial_y u = -\partial_n u = 0 \]

However, this is not justified rigorously.

Near the separation point, the structure of the flows can be described by the so-called *triple deck theory* (See Childress Sec. 8.4). This separation yields the drag on the solid body (see C.M. P81).

At the point of separation, the Prandtl boundary layer equations are not good. With the separation of boundary layers, we understand why vortex shedding and vortical wakes exist when we resolve the D’Alembert paradox.
5 Matching with the outer solution? Difficulties

The inner solution is given by Prandtl's equations and the outer solution is given by the Euler equations. Hopefully, the matching of them could give an approximated solution to the Navier-Stokes equations for small $\mu$ or large $Re$.

The issue is how to match these two solutions. There are some strategies but some difficulties are there. Check C.M. P79 for details.