Math 575-Lecture 15

In this lecture, we look at a thin film of viscous fluid between two solid surfaces. This then leads to the so-called lubrication theory. **We will ignore gravity.**

The dominated equations are the dimensionless NS equations.

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u, \\
\nabla \cdot u &= 0, \\
\nu &= \frac{1}{Re}, \\
u_0 &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

where \( \nu = \frac{1}{Re} \).

1 The setup

Consider a given surface \( z = h(x, y, t) \) above a solid plate \( z = 0 \). The region between these two surfaces is filled with a viscous Newtonian fluid.

We assume the scale length in the \( x, y \) dimension is \( L \) while \( H \) is the length scale for \( z \). \( H \gg L \).

\[ u = \langle u, v, w \rangle. \]

Assume the typical velocity for \( u \) and \( v \) are \( U \). Then, due to the incompressibility,

\[ u_x + v_y + w_z = 0 \]

we find the scale for \( w \) is

\[ W \sim UH/L \ll U. \]

The velocity in the \( z \) dimension is very small but the variation in \( z \) derivative is large.

1.1 Relative importance within different terms

Now, we do scaling to figure out which terms are important and which are not.

We do scaling

\[ t' = t/T, \quad x' = x/L, \quad y' = y/L, \quad z' = z/H, \quad u' = u/U, \quad v' = v/U, \quad w' = w/(UH/L), \quad p' = p/P. \]

The incompressibility condition is reduced to

\[ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0. \]

Hence, all terms are equally important—though \( w \) is small, the variation is big and the derivative is comparable with others.
The conservation of momentum in $x$ direction is given by

$$\frac{U}{T} \partial_t u' + \frac{U^2}{L} (u' \partial_x u' + v' \partial_y u' + w' \partial_z u') = -\frac{P}{L} \partial_x p' + \nu \frac{U}{H^2} \Delta_{x'y'} u' + \frac{\partial^2 u'}{\partial z'^2}$$

We assume the Reynolds number for $u, v$,

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \gg 1,$$

but the Reynolds number for the $z$ direction

$$Re_z = \frac{WH}{\nu} = Re \frac{H^2}{L^2} \ll 1.$$

Hence, the important terms will be $\partial_z^2 u$ which must be balanced by $\partial_z p$. Consequently, the leading order terms in the dimensional form are given by

$$0 = -\partial_x p + \nu \frac{\partial^2}{\partial z^2} u$$

This also yields the scale for $p$ as $P = \nu UL / H^2$.

Similarly,

$$0 = -\partial_y p + \nu \frac{\partial^2}{\partial z^2} v.$$

Lastly, for the transverse direction, we have

$$\frac{UH}{TL} \partial_t w' + \frac{U^2 H}{L^2} (u' \partial_x w' + v' \partial_y w' + w' \partial_z w') = -\frac{P}{H} \partial_x p' + \nu \frac{U}{HL} \Delta_{x'y'} w' + \frac{\partial^2 w'}{\partial z'^2}$$

Clearly, $P/H$ are much larger than other scales in the equation. Hence,

$$\partial_z p = 0$$

The system of equations are called the lubrication approximation

$$0 = -\partial_x p + \nu \frac{\partial^2}{\partial z^2} u,$$

$$0 = -\partial_y p + \nu \frac{\partial^2}{\partial z^2} v,$$

$$0 = -\partial_z p.$$
2 Reynolds Lubrication equation

By the third equation, we find

\[ p = p(x, y, t) \]

Now, we assume the boundary condition at \( z = h(x, y, t) \) is given by

\[ u = \langle A(x, y, t), B(x, y, t), C(x, y, t) \rangle \]

then, the no-slip condition yields,

\[ C(x, y, t) = h_x A(x, y, t) + h_y B(x, y, t) + h_t. \]

By the first two equations, we solve

\[ u = \frac{p_x}{2\nu}(z^2 + D_1 z + D_2), \]
\[ v = \frac{p_y}{2\nu}(z^2 + D_3 z + D_4) \]

Using the no-slip conditions like \( u|_{z=0} = 0 \) and \( u|_{z=h} = A \), we find

\[ u = \frac{p_x}{2\nu}(z^2 - zh(x, y, t)) + \frac{Az}{h}, \]
\[ v = \frac{p_y}{2\nu}(z^2 - zh(x, y, t)) + \frac{Bz}{h}. \]

Now, integrating the incompressible condition

\[ \partial_x u + \partial_y v + \partial_z w = 0 \]

for \( z : 0 \to h \), one obtains that

\[ \int_0^h u_z dz = \partial_x \int_0^h u dz - Ah_x, \]
\[ \int_0^h v_y dz = \partial_y \int_0^h v dz - Bh_y, \]
\[ \int_0^h w_z dz = C \]

We find

\[ \partial_x \int_0^h u dz + \partial_y \int_0^h v dz + h_t = 0. \]
Further,
\[ \bar{u} = \int_0^h u\,dz = \frac{p_x}{12\nu}h^3 + \frac{A}{2}h, \]
\[ \bar{v} = \int_0^h v\,dz = \frac{p_y}{12\nu}h^3 + \frac{B}{2}h. \]

Consequently, we have
\[ \nabla \cdot (h^3\nabla p) = 6\nu(h_t + C + A_xh + B_yh). \]

### 3 Discussions and examples

#### 3.1 Hele-Shaw flows

If \( A = B = C = 0 \) and \( h \) is a constant, we find that
\[ \Delta p = 0. \]

Hence, the averaged velocities satisfy
\[ \langle \bar{u}, \bar{v} \rangle = \frac{h^3}{12\nu}\nabla p. \]

This means that the viscous thin fluid can be regarded as an 2D inviscid harmonic flow.

Such kind of flows are called Hele-Shaw flows. This makes the friction between the two rigid surface vanishingly small.

#### 3.2 Slider bearing

Consider a finite plate with slope \( \alpha = \tan \theta \ (\alpha \ll 1) \), so that its initial shape is described by \( h = h_1 + \alpha x \) the two ends are at \( h = h_1 \) and \( h = h_2 \) or \( x = 0, x = (h_2 - h_1)/\alpha \).

Now, assume the plate is moving with velocity \( U\hat{x} \) and then \( A = U_0, B = C = 0 \). Consequently,
\[ h = h_1 + \alpha(x - U_0t) \]

Then,
\[ \partial_x(h^3p_x) = -6\nu\alpha U_0 \]

If we rewrite \( P(h, t) = p(x(h, t), t) \), then, we have
\[ (h^3P)_h = -6\nu U_0/\alpha. \]
Assume that at the two ends of the plate, $p = p_0$. Then, we can solve that

$$P = p_0 + \frac{6\nu U_0}{\alpha(h_1 + h_2)h^2}(h - h_1)(h_2 - h)$$

One can argue that $\sigma = -pI + 2\mu E \sim -pI$. Consequently, the pressure dominates the force acting on the plate. The lift is given by

$$\int_{h_1}^{h_2} \frac{1}{\alpha} (P - p_0) dh = \frac{6\mu U_0}{\alpha^2(h_1 + h_2)} (-2(h_2 - h_1) + (h_1 + h_2) \ln(h_2/h_1))$$

The drag is much less than the lift.

Suppose the total length of the plate is $L$. Then, $\alpha = (h_2 - h_1)/L$. Inserting this into the formula, we find that the left becomes very large as $\alpha$ becomes very small.

Hence, the two plates can slide smoothly without much friction. However, if $U_0 > 0$, we need a large force to keep them together. If $U_0 < 0$, we need a large force to keep them apart. It is very hard to separate them if $U_0 < 0$.

### 3.3 Last comment

Above, we have assumed that

$$Re = Re \frac{H^2}{L^2} \ll 1.$$ 

If this is not valid, then the theory breaks up.

Actually, later, in the boundary layer theory, this is not valid any more.