Math 575-Lecture 14

We continue to study some properties of N-S equations and look at some examples of viscous flows.

1 Viscous dissipation of energy

Consider the kinetic energy

\[ E = \frac{1}{2} \int_{R_t} \rho |\mathbf{u}|^2 dV \]

where \( R_t \) is a material volume of fluid.

Then, by the second convection theorem

\[
\frac{d}{dt} E = \frac{1}{2} \int_{R_t} \rho \frac{D}{Dt} |\mathbf{u}|^2 dV = \int_{R_t} \rho \mathbf{u} \cdot \frac{D}{Dt} \mathbf{u} dV = \int_{R_t} \mathbf{u} \cdot (-\nabla p + \mu \Delta \mathbf{u} + \rho \mathbf{b}) dV
\]

\[
= \int_{R_t} -\nabla \cdot (\mathbf{u} p) dV + \mu \int_{R_t} \mathbf{u} \Delta \mathbf{u} dV + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{u} dV = \int_S \mathbf{u} \cdot (-p\mathbf{n} + \mu \mathbf{n} \cdot \nabla \mathbf{u}) dS - \mu \int_{R_t} |\nabla \mathbf{u}|^2 dV + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{u} dV
\]

In the last step, we did the following:

\[
\int_{R_t} u_j \partial_i^2 u_j dV = \int_{R_t} \partial_i (u_j \partial_i u_j) dV - \int \partial_i u_j \partial_i u_j dV = \int_S n_i u_j \partial_i u_j - \int \partial_i u_j \partial_i u_j dV
\]

This is some energy estimate that may be useful if \( \mathbf{u} = 0 \) on \( S \), but it does not have enough physical meaning.

Now, consider the fact

\[ 0 = \int_{R_t} \mu \mathbf{u} \cdot \nabla \cdot (\nabla \mathbf{u}^T) dV = \int_{R_t} \mu u_i \partial_j (u_j \partial_i u_j) dV = \int_S \mu u_i n_j \partial_i u_j dS - \int_{R_t} \partial_i u_j \partial_i u_j dV \]

Adding this to the equation, we have

\[
\frac{d}{dt} E = \int_S \mathbf{u} \cdot (-p\mathbf{n} + \mathbf{n} \cdot 2\mu E) dS - \mu \int_{R_t} \partial_i u_j (\partial_i u_j + \partial_j u_i) dV + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{u} dV
\]

\[
= \int_S \mathbf{u} \cdot (\mathbf{n} \cdot \sigma) dS - 2\mu \int_{R_t} \nabla \mathbf{u} : \mathbf{E} dV + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{u} dV
\]

Note \( \nabla \mathbf{u} : \mathbf{E} = E : E \) and thus we have

\[
\frac{d}{dt} E = \int_S \mathbf{u} \cdot (\mathbf{n} \cdot \sigma) dS - 2\mu \int_{R_t} |\mathbf{E}|^2 dV + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{u} dV
\]

The first term is the work done by hydrodynamics force; the second term represents the energy dissipated and therefore it is the dissipation rate; the last term means the work done by body force.
2 The scaling invariance

If we take the scaling for $\Omega = \mathbb{R}^d$ as $x = \lambda \bar{x}$, $t = \lambda^{\alpha+1} \bar{t}$, then, by the physical meaning

$$u = \frac{dx}{dt} = \lambda^{-\alpha} u \Rightarrow u_\lambda = \lambda^\alpha u(\lambda \bar{x}, \lambda^{\alpha+1} \bar{t}). \quad (2.1)$$

With the scaling $p_\lambda = \lambda^{2\alpha} p(\lambda \bar{x}, \lambda^{\alpha+1} \bar{t})$, we find

$$\partial_\bar{t} u_\lambda + u_\lambda \cdot \nabla_{\bar{x}} u_\lambda + \nabla_{\bar{x}} p_\lambda = \lambda^{2\alpha+1} (\partial_t u + u \cdot \nabla u + \nabla p)$$

The viscosity term is scaled as

$$\Delta_{\bar{x}} u_\lambda = \lambda^{\alpha+2} \Delta u$$

Hence, the equation is scaling-invariant if

$$\alpha = 1. \quad (2.2)$$

**Theorem 1.** Let $\Omega = \mathbb{R}^d$. If $(u, p)$ is a solution, then $(\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t))$ is also a solution.

A self-similar solution will be of the form (by choosing $\lambda = t^{-1/2}$)

$$u(x, t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$$

3 Stress in other coordinates

The stress vector is

$$t = n \cdot \sigma = -pn + 2\mu n \cdot E = -pn + \mu [2(n \cdot \nabla)u + n \times (\nabla \times u)].$$

One may use this to compute the stress vector. Alternatively, one may compute $E$ and then compute the stress.

**Example.** Consider the flow outside a rotation circular cylinder of the form

$$u = v(r) \hat{\theta}$$

Find the stress vector acting on the fluid at the surface of the cylinder.

Direct computation shows

$$\nabla u = (\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \partial_\theta + \hat{z} \partial_z) [v(r) \hat{\theta}] = \hat{r} \frac{\partial v(r)}{\partial r} \hat{\theta} + \frac{1}{r} \hat{\theta} \otimes v(r) \partial_\theta \hat{\theta} = v'(r) \hat{r} \otimes \hat{\theta} - \frac{v(r)}{r} \hat{\theta} \otimes \hat{r}.$$
Hence,
\[ 2E = \nabla u + \nabla u^T = \left( v'(r) - \frac{v(r)}{r} \right) (\hat{r} \otimes \hat{\theta} + \hat{\theta} \otimes \hat{r}). \]

At the surface, the normal of the surface is \( n = -\hat{r} \). Then,
\[ t = p\hat{r} - \mu \left( v'(r) - \frac{v(r)}{r} \right) \hat{\theta} = p\hat{r} - \mu r \frac{d}{dr} \left( \frac{v}{r} \right) \hat{\theta}. \]

4 Some examples of incompressible viscous flows

In this section, we look at some examples. Below, we consider constant density so that we introduce \( \nu = \mu/\rho \).

4.1 The Couette flow

Two rigid planes at \( y = 0, H \). \( y = 0 \) plane is stationary while \( y = H \) plane is moving with constant velocity \( U\hat{x} \). Solve the N-S equation in between these two planes.

Due to symmetry, all quantities depend on \( y \) only. We assume \( u = \langle u(y), 0 \rangle \).

By the equation
\[ 0 = u \cdot \nabla u = -\nabla p + \mu \Delta u \]

We find
\[ p_y = 0, \ 0 = \mu \Delta u(y) \]

so \( p \) is a constant and \( u_{yy} = 0 \) and \( u(y) = \frac{Uy}{H} \). We have shear flow.

The viscosity stress \( \tau \) is nonzero.

4.2 Rayleigh problem

Consider the unsteady problem: the fluid is in the region \( y > 0 \). \( y = 0 \) is moving in \( x \) direction with velocity \( U(t) = U_0 \cos \omega t \).

By symmetry, one may assume
\[ u = \langle u(y, t), 0 \rangle \]

Similar as above, we find
\[ p = \text{const}, \ u_t - \nu u_{yy} = 0 \]

Now, the equation for \( u \) is linear and we can assume (Fourier transform)
\[ u(y, t) = \text{Re}(f(y)e^{i\omega t}) \]
Then,
\[ i \omega f - \nu f_{yy} = 0, \quad f(0) = U_0 \]

Hence,
\[ u(y, t) = U_0 \cos \left( \omega t - \sqrt{\frac{\omega}{2\nu}} y \right) e^{-y\sqrt{\omega/(2\nu)}} \]

The velocity dies away quickly. To be more precise, after a layer of thickness \( O(\nu^{1/2}) = O(Re^{-1/2}) \), the effect of the moving plate dies away.

### 4.3 Poiseuille flow

A Newtonian viscous fluid is flowing inside a cylinder tube with radius \( R \) in steady motion. Let the axis be the \( z \)-axis. We then by symmetry have

\[ \mathbf{u} = u(r) \hat{z}, \quad p = p(r, z) \]

The equations
\[ \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \Delta \mathbf{u} \]

read in cylindrical coordinates as

\[ 0 = -\left( \hat{r} \partial_r p + \hat{z} \partial_z p \right) + \mu \left( \frac{1}{r} \partial_r (r \partial_r (u(r) \hat{z})) + \frac{1}{r^2} \partial^2_{\theta}(u(r) \hat{z}) + \partial^2_z (u(r) \hat{z}) \right) \]

Hence, we must have \( \partial_r p = 0 \) and \( p = p(z) \).

Also
\[ 0 = -\partial_z p + \mu \frac{1}{r} (ru')' \]

Separation of variables, we have \( \partial_z p = -G = \text{const} \) and

\[ u'' + \frac{1}{r} u' = -\frac{G}{\mu} \]

The general solution is given by (the homogeneous equation is the Euler ODE)

\[ u(r) = C_1 \log r + C_2 - \frac{G}{4\mu} r^2 \]

Clearly, \( C_1 = 0 \) and \( C_2 = \frac{G}{4\mu} R^2 \) by the no-slip boundary condition.

The importance of Poiseuille flow is that it can be used to measure the viscosity by measuring the pressure needed and the flux.
4.4 Stagnation Point flow

Consider a 2D steady viscous flow in \( y > 0 \) plane whose streamfunction is given by

\[
\psi(x, y) = UL^{-1}xF(y)
\]

The vorticity is given by

\[
\omega = -\Delta \psi = -UL^{-1}xF''(y)
\]

The N-S equations in vorticity form read

\[
u \cdot \omega - \nu \Delta \omega = 0
\]

where \( \nu = \mu/\rho \).

Then,

\[
F'F'' - FF''' - Re^{-1}F'''' = 0
\]

where \( Re = UL/\nu \).

The boundary condition: \( u = 0, y = 0 \). Hence, \( \psi_y = 0 \) or \( F'(0) = 0 \) and \( \psi_x = 0 \) or \( F(0) = 0 \).

If one aims to obtain a stagnation point flow so that \( \psi \sim UL^{-1}xy \) at infinity, then, \( F(y) \sim y \) as \( y \to \infty \). Then, as \( y \to \infty \), \( F'(y) \to 1 \) and \( F''(y) \to 0, F'''(y) \to 0 \). Hence, integrating once, we have

\[
(F')^2 - FF'' - Re^{-1}F''' = 1
\]

Doing the scaling \( \eta = Re^{1/2} \) and \( F(y) = Re^{-1/2}f(\eta) \), one have

\[
(f')^2 - ff'' - f''' = 0
\]

for \( f(\eta) \). \( f(\eta) \) satisfies \( f(0) = f'(0) = 0 \) and \( f'(\infty) = 1 \).

Then, after \( \eta = 1 \), \( f' \sim 1 \). This means the velocity transits from 0 to 1. This layer is of thickness \( O(Re^{-1/2}) \), which is the typical length of boundary layer.