Practice problems

1. Set up the integral without evaluation. The volume inside \((x - 1)^2 + y^2 + z^2 = 1\), below \(z = \sqrt{3}r\) but above \(z = -r\).

   This problem is very tricky in cylindrical or Cartesian since we must divide the region into several blocks if we use these two. It is convenient to use spherical. \(\pi/6 \leq \phi \leq 3\pi/4\) by \(z = \sqrt{3}r\) and \(z = -r\). The sphere is \(\rho = 2 \sin \phi \cos \theta\). The range for \(\theta\) is from \(-\pi/2\) to \(\pi/2\). This is because the sphere is on the right of \(yz\) plane. At the starting point \(\rho = 0\) and \(\cos \theta = 0\). This means it starts from \(-\pi/2\). Lastly, \(0 \leq \rho \leq 2 \sin \phi \cos \theta\).

2. Set up the integral for the moment of inertia about \(z\) axis inside both \(\rho = 2\) and \(r = 2 \cos \theta\), outside \(r = 1\) and above \(xy\) plane. The density is \(\delta = \sqrt{x^2 + y^2 + z^2}\).

   This problem is good for cylindrical coordinates. \(-\pi/3 \leq \theta \leq \pi/3\) and \(1 \leq r \leq 2 \cos \theta, 0 \leq z \leq \sqrt{4 - r^2}\).

3. Set up the integral for the volume outside \(r = 1\) inside \(\rho = 2\) in both cylindrical and spherical coordinates.

   The tricky part is the spherical. In spherical, the two intersects at \(2 \sin \phi = 1\) or \(\pi = \pi/6, 5\pi/6\). Hence, in spherical, \(0 \leq \theta < 2\pi, \pi/6 \leq \phi \leq 5\pi/6\). For the \(\rho\) part, \(r = 1\) is \(\rho \sin \phi = 1\). Hence, \(1/\sin \phi \leq \rho \leq 2\).

1. Divergence, curl

   (a) In the daytime under sunshine, the algae in an ocean generate oxygen (they also consume oxygen but the net effect is oxygen production). The rate of production is clearly proportional to the density of algae in the ocean. Suppose that the oxygen consumption by other plants and animals in the ocean can be neglected and that the density distribution of the oxygen reaches equilibrium. The equilibrium is kept under a flow of the oxygen which is the result of diffusion, transportation etc. We model the ocean by \(\mathbb{R}^3\) and the field of the oxygen flow is given by

   \[
   \mathbf{F} = \langle \arctan(x) + e^{-y^2}, \frac{x^2}{1 + z^2 + x^4} + \arctan(y), \arctan(z) + \frac{\sin^4(x)}{y^4 + 1} \rangle.
   \]
If the density of the algae at \((0, 0, 0)\) is \(3 \times 10^3\) per cubic centimeter, what is the density at \((1, 1, 1)\)?

This is a quiz problem.

(b) Suppose there is a cloud of charged dust. The electronic field generated by the dust is given by

\[ \mathbf{E} = (xy^2, xyz, \sin(x)). \]

What is the charge density at \((0, 0, 0)\)? If the dust is positively charged at \((1, 1, 1)\) with \(3 \times 10^{-6}\text{C/cm}^3\), what is the charge density at \((-1, -1, 2)\)? Is it negatively charged or positively charged?

(c) In a storm weather, near \((\pi/2, 1, 1)\), the velocity field of the air was roughly given by

\[ \mathbf{v} = (xy^2, xyz, \sin(x)). \]

What is the vorticity at \((\pi/2, 1, 1)\)?

2. Line integrals

(a) Parametrization

- Parametrize \(x^2 + 4y^2 = 1\) \(r(t) = (\cos t, \frac{1}{2} \sin t), 0 \leq t < 2\pi\)
- Parametrize the boundary of the region bounded by \(x\)-axis, \(y = \sqrt{2}x\), and \(x = 1\). \(C = C_1 + C_2 + C_3. C_1 : r(t) = (t, 0), 0 \leq t \leq 1. C_2 : r(t) = (1, t), 0 \leq t \leq 1. C_3 : r = (t, \sqrt{t}), t : 1 \to 0\)
- Parametrize the ellipse formed by the intersection of \(x^2 + y^2 = 1\) and \(x + z = 0. \)
  \[ r(t) = (\cos t, \sin t, -\cos t) \]

(b) Usual line integrals (2 types)

- Consider the curve \(x^2/4 + y^2 = 1\) with \(x \geq 0, 0 \leq y \leq 1/2\). If the density (per unit length) is \(\delta = y/x\), compute the moment of inertia \(I_y\).
  \[ I_y = \int_C x^2 \delta ds = \int_C x y ds. \quad r = (2 \cos t, \sin t). 0 \leq t \leq \pi/6 \]
- Compute the line integral of \(\mathbf{F} = (3y, -2x)\) over the curve \(y = x^2\) for \(0 \leq y \leq 1\) oriented from right to left.
- Let \(r = (t^3, t^2, t), 0 \leq t \leq 1. \) Compute \(\int_C \mathbf{F} \cdot \mathbf{T} ds\) where \(\mathbf{F} = (e^{y^2}, 0, ye^{y^2})\)

(c) Conservative field.
• Let \( \mathbf{r}(t) = (\ln(1 + t^9), t^3 + 1, t^{100}) \), \( 0 \leq t \leq 1 \). Compute \( \int_C x \, dy + y \, dx + z \, dz \)

The field is \( \nabla (xy + z) \). Or you can notice that it is \( \int d(xy + z) \)

• \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \) where \( \mathbf{F} = \langle x^e x^z + e^x, 2yz, x e^x + y^2 \rangle \). \( \mathbf{r} = (e^{t^2}, e^{t^3}, t^4) \). \( t \in [0, 1] \)

It is irrotational and thus conservative. \( \phi = e^{xz} + y^2 z \)

3. Green’s theorem

(a) Circulation and flux

• Compute the line integral of \( \langle x^2 y + x e^{x^3}, x y - \sin^2(e^y) \rangle \) over the rectangle with vertices \((0, 0), (2, 0), (0, 3) \) and \((2, 3) \) oriented counterclockwise.

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint P \, dx + Q \, dy,
\]

\( P = x^2 y + x e^{x^3}, Q = x y - \sin^2(e^y) \).

Then, \( Q_x - P_y = y - x^2 \).

\[
\int_0^3 \int_0^{x^2} (y - x^2) \, dx \, dy
\]

• Compute the line integral \( \oint_C P \, dx + Q \, dy \) where \( P = x y, Q = x^2 \) and \( C \) is the loop of the curve in the first quadrant whose polar equation is \( r = \sin(2\theta) \).

We have \( Q_x - P_y = 2x - x = x \). Hence, \( \iint_D x \, dA \). In polar, \( 0 \leq \theta \leq \pi/2, 0 \leq r \leq \sin(2\theta) \)

\[
\int_0^{\pi/2} \int_0^{\sin(2\theta)} r \cos \theta rdrd\theta
\]

The integral \( \int_0^{\pi/2} \sin^3(\theta) \cos^4(\theta) \, d\theta \) can be done by doing substitution \( u = \cos \theta \)

• Let \( \mathbf{F} = xy^2 \mathbf{i} + x^2 y \mathbf{j} \). Let \( C \) be the union of \( x \)-axis(\( |x| \leq 2 \)) and \( y = \sqrt{4 - x^2} \), oriented counterclockwise. Compute the flux \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \) in two ways.
The first way is Green’s theorem \( \oint F \cdot n \, ds = \iint_D \nabla \cdot F \, dA \).

Then, you have \( \iint_D (x^2 + y^2) \, dA \) which can be finished using polar.

The second way is to parametrize directly. \( n \, ds = \langle dy, -dx \rangle \).

Then, \( \oint P \, dy - Q \, dx \).

We parametrize the boundary with two pieces.

- If \( v = \langle y^2, xy \rangle \), and \( C \) is the ellipse \( x^2/9 + y^2/4 = 1 \), compute the circulation \( \oint F \cdot \, dr \) in two ways.

- \( R \) is the region inside \( (x-1)^2 + y^2 = 1 \) but outside \( x^2 + y^2 = 1 \). Let \( C \) be the boundary of \( R \) oriented counterclockwise. Let \( F = \langle e^{2x^2-3x} + y, y \rangle \). Let \( C_1 \) be part of \( C \) which is on \( x^2 + y^2 = 1 \) and \( C_2 \) be the part of \( C \) which is on \( (x-1)^2 + y^2 = 1 \). Compute \( \int_{C_2} F \cdot T \, ds \) by computing the circulation on \( C \) using Green’s and \( \int_{C_1} F \cdot \, dr \) (Hint: \( \int_{C_1} e^{2x^2-3x} \, dx = 0 \) by the Fundamental theorem).

(b) Find the area of the region enclosed by \( r(t) = \langle \sin(2t), \sin(t) \rangle \) above \( x \)-axis.

For the loop, \( 0 \leq t \leq \pi \). Then, \( A = -\oint y \, dx = -\int_0^\pi \sin(t)2 \cos(2t) \, dt \).

\( u = \cos t \) and \( \cos(2t) = 2 \cos^2 t - 1 = 2u^2 - 1 \). Answer is 4/3

- Find the area of the region enclosed by one arch of the cycloid \( x(t) = a(t - \sin(t)), y(t) = a(1 - \cos(t)) \) and \( x \)-axis.

Example in class

(c) Suppose \( v = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle \). Suppose \( C \) is a simple closed curve in the plane, oriented counterclockwise. What values can \( \oint_C v \cdot n \, ds \) be? (Hint: This field is divergence free.)

Since it is divergence free except at the origin, we discuss two cases. If \( C \) doesn’t enclose the singular point, the answer is zero by Green’s. If the curve encloses the origin, then we use a small circle to isolate the origin. The flux then equals the flux on the small circle. The answer is \( 2\pi \). Then, the answer could be 0 or \( 2\pi \)

- Suppose \( F = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \). Suppose \( C \) is a simple closed curve in the plane, oriented counterclockwise. What values can \( \oint_C F \cdot T \, ds \) be? (Hint: This field is curl free or \( P_y = Q_x \).)

4. Surface integrals (2 types)

(a) Parametrization, and the surface integral of a function
• Consider the surface of revolution obtained by revolving \( x = f(z) \) about \( z \) axis. Parametrize this surface. Use \( z \) and \( \theta \) as the parameters. 
\[
  x = f(z) \cos \theta, y = f(z) \sin \theta, z = z
\]

• Consider the fence \( S: x = 2 \sin(t), y = 8 \cos(3t), 0 \leq t < 2\pi \) and \( 0 \leq z \leq 2 \). Set up the surface integral \( \iint_S 2yz \, dS \). Use \( t, z \) for the parameters. 
\[
  \mathbf{r} = \langle 2 \sin t, 8 \cos(3t), z \rangle.
\]

(b) Flux

• Compute the flux of \( \mathbf{G} = \langle 2x, x - y, y + z \rangle \) through the surface \( S \), which is the portion of the plane \( 2x - 3y + 5z = 0 \) inside \( x^2 + y^2 = 1 \) oriented upward. The surface has a boundary and the field is not a curl (the divergence is nonzero). We parametrize the surface 
\[
  \mathbf{r} = \langle x, y, \frac{1}{5}(3y - 2x) \rangle, x^2 + y^2 \leq 1.
\]
Then, the flux is 
\[
  \Phi = \iint_D \mathbf{G} \cdot (\mathbf{r}_x \times \mathbf{r}_y) \, dxdy
\]
Note that \( \mathbf{r}_x \times \mathbf{r}_y \) always points up for a function graph and we are fine. \( D \) is the disk. For this problem, use polar. Don’t forget to write \( z \) in terms of \( x, y \)

• Parametrize the upper hemi-sphere with radius 1. Then compute the flux of \( \mathbf{F} = \langle x^2, 0, 0 \rangle \) across it. 
\[
  \mathbf{r} = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle. \quad 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi, 0 \leq \rho \leq 1
\]

• \( \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \) where \( \mathbf{F} = \langle y, -x, z \rangle \) and \( S \) is the surface \( z = \theta, 0 \leq \theta \leq \pi \) and \( 1 \leq x^2 + y^2 \leq 4 \). Parametrize the surface. 
\[
  \mathbf{r} = \langle r \cos \theta, r \sin \theta, \theta \rangle, 1 \leq r \leq 2, 0 \leq \theta \leq \pi.
\]
Then, \( \mathbf{n} \, dS = \mathbf{r}_r \times \mathbf{r}_\theta \, dr \, d\theta \)

• Compute the flux of \( \mathbf{F} = \langle 2, 2, 3 \rangle \) across the surface \( S: \mathbf{r}(u, v) = \langle u + v, u - v, uv \rangle, 0 \leq u, v \leq 1 \)
We compute directly that \( \mathbf{n} \, dS = \mathbf{r}_u \times \mathbf{r}_v \, du \, dv = \ldots \)

• Transform the integrals into double integrals without evaluating:
  - \( \iint_S (z^2 - y^2) \, dS \) and \( \iint_S (z^2 - y^2) \, dxdy \). \( S \) is the part of \( x^2 + y^2 \) on the left of \( x = 9 \).
  - \( \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \). \( \mathbf{F} = \langle x, y, z \rangle \) and \( S \) is the cone \( z = r \) between \( z = 2 \) and \( z = 5 \), with the normal pointing downward.

5. Stokes Theorem
• Let $F = \nabla \times (yz, -xz, z^2)$ and $S$ is $4x^2 + y^2 = e^z$ between $z = 0$ and $z = 2\ln 2$, with the normal point upward. Compute

$$\iint_S F \cdot n \, dS$$

• Let $F = \vec{i} + x\vec{j} - y\vec{k}$. Let $C$ be the intersection between $x^2 + y^2 = 1$ and $z + y = 3$ oriented counterclockwise if viewed from above. Compute $\oint_C F \cdot T \, ds$ in two ways.

The first way is direct computation by parametrization. $r(t) = \langle \cos t, \sin t, 3 - \sin t \rangle$.

For the second way, we can apply Stokes’s theorem. $\iint_S \nabla \times F \cdot n \, dS$. Then, we have to parametrize the surface $r = \langle x, y, 3 - y \rangle$.

If the vector field is $F = \langle z^3 + \sin(x), x + e^{y^2} + e^y, -y - \cos(8z) - z^{100} \rangle$, can you reduce this to the problem here? Why?

Yes. The extra field $G = \langle x^3 + \sin(x), e^{y^2} + e^y, -\cos(8z) - z^{100} \rangle$ is conservative. The integral on the closed curve is zero.

• Let $F = \langle 3y, -2x, x^2y^2z^2 \rangle$. Let $S$ be the surface $x^2 + y^2 + (z - 1)^2 = 2$ above $xy$ plane, oriented upward. Compute

$$\iint_S (\nabla \times F) \cdot n \, dS$$

Apply Stokes and reduce it to the circulation over the curve $x^2 + y^2 + (0 - 1)^2 = 2$. On this curve, the field is $F = \langle 3y, -2x, 0 \rangle$. The curve is $r = \langle \cos t, \sin t, 0 \rangle$. The computation is easy.

• Suppose $F = \langle xy^2z^2 + y, x^2yz^2 + z, x^2y^2z + x \rangle$. $C$ is the hexagon with vertices $(2, 0, 1), (1, 0, 2), (0, 1, 2), (0, 2, 1), (1, 2, 0)$ and $(2, 1, 0)$, which are all in the plane $x + y + z = 3$. Compute the circulation of the field over this curve. (Hint: The best way is to split a conservative field first. Anyway, using Stokes theorem directly will give you the same answer.)

Apply Stokes. $\nabla \times F = \nabla \times \langle y, z, x \rangle$ because the extra field is $\nabla(x^2y^2z^2/2)$ which has zero curl. If you don’t notice this, it is fine since it will be canceled as well. Then, the order is clockwise.

$$-\iint_D (-1, -1, -1) \cdot (1, 1, 1) \, dx \, dy = 3 \text{Area}(D) = 9$$

• Compute the line integral of $\vec{G} = \langle x^2 - 2y, 2e^y - z, z^3 + 3x \rangle$ along the curve $r(t) = \langle \cos t, \sin t, \cos t \sin t \rangle$ where $t : 0 \to 2\pi$.

Note that the curve is closed since $r(0) = r(2\pi)$. Then, we can throw away a conservative field and have $\langle -2y, -z, 3x \rangle$. At
this point, you can either use direct computation or use Stokes’s theorem. If we apply Stokes’s, noticing that the curve is on \( z = xy \). Hence, the surface is \( r = \langle x, y, xy \rangle \), \( x^2 + y^2 \leq 1 \). We have \( \iint_D (2 + 3x - y) \, dx \, dy \). However, \( x^2 + y^2 \leq 1 \) is the disk. The integral of \( x, y \) are zero. Then, \( 2 \times \text{Area}(D) = 2\pi \)

6. Divergence theorem

(a) Computation and applications

- Let \( a = \langle 2, -3, 4 \rangle \), \( b = \langle -1, 1, 2 \rangle \) and \( c = \langle 2, 3, -5 \rangle \). Let \( T \) be the parallelepiped determined by \( 2a + b, b, 3b - 2c \). \( S \) is the boundary of \( T \) with outer normal \( n \). Compute

\[
\iint_S F \cdot n \, dS,
\]

where \( F = \langle x^2 - 1, y - 2xy + e^{8z}, 3z + \ln(1 + x^2) \rangle \).

- Compute the flux \( \iint_S F \cdot n \, dS \) where \( F = \langle y^3 + z^2, xy - xz^2, xe^y \rangle \). \( S \) is the boundary of the solid \( x + y + z \leq 1 \), \( x, y, z \geq 0 \), with \( n \) being the outer normal.

By divergence, we have \( \iiint_T (0 + x + 0) \, dV \). This volume is \( 0 \leq x \leq 1, 0 \leq y \leq 1 - y, 0 \leq z \leq 1 - x - y \). Set up the volume integral and integrate.

- Let \( F = \langle x + e^{8z^2}y^2, y + 3y^2 + \ln(x^8 + 1000), z + \cos(xy) \rangle \). Let \( T \) be the upper hemi-ball with radius 1. Compute the flux of this field out of \( T \).

By divergence theorem, \( \iiint_T (1 + 1 + 6y + 1) \, dV \). The integral of \( y \) is zero. Hence, \( 3\text{Vol}(T) = 3 \times \frac{2}{3} \pi 1^3 = 2\pi \)

- Suppose \( \mathbf{v} \) is the gravitational field generated by a cloud of mass. The physical law tells us that \( \iint_S \mathbf{v} \cdot n \, dS = -4\pi G m \) where \( m \) is the total mass inside \( S \). Suppose the density of the mass is \( \delta \). Then, the total mass is \( m = \iiint_T \delta \, dV \). Using the divergence theorem, show that \( \delta = -\frac{1}{4\pi G} \nabla \cdot \mathbf{v} \).

(b) Surface independence

- Consider the surface \( z = (x^2 + y^2 - 1)(x^4 + y^4 + 1) \) for \( z \leq 0 \), oriented upward. Compute \( \iint_S F \cdot n \, dS \) where \( F = \langle xy^2 + y^2 z, x^2 z - x^2 y, 3z - xy^2 \rangle \).

Divergence free. We use surface independence. The surface we pick then is the one in \( z = 0 \) plane. Hence, it is the region inside \( x^2 + y^2 - 1 = 0 \). We find that \( F \cdot k = 0 \). The final answer is therefore zero.
• let \( S \) be \( r(t, z) = \langle (1 - z)^3 \cos t, (1 - z)^2 \sin t, z \rangle \) and \( 0 \leq t < 2\pi, 0 \leq z \leq 1 \). The normal is upward. Compute the flux of \( \vec F = \langle y^2 z - z^2, 4 - xy, 3 + xz \rangle \) through this surface.

The surface is a surface with boundary \( r(t, z = 0) = \langle 3 \cos t, 2 \sin t, 0 \rangle \).

The field is divergence free. Hence, we can apply the surface independence technique and have the flux to be equal to the flux through the part inside \( r(t, z = 0) \) in \( xy \) plane. Hence, \( \iint_D \vec F \cdot \vec k dA = \iint_D 3 dA \). The area can be computed using whatever way you want. You can either use the line integral or use double integrals by change of variable. If you notice it is an ellipse, you can get the answer quickly as well.

• Compute \( \iint_S \vec F \cdot \vec n dS \) where \( \vec F = \langle y^2 z - z^2, 4 - xy, 3 + xz \rangle \) and \( S \) is the surface of revolution by revolving \( z = -x^2 + 3x - 2, z \geq 0 \) about \( z \) axis and the normal is downward.