12.4: Lines and planes in space

Suppose \( P(x, y, z) \) is an arbitrary point on the geometric objects such as lines and surfaces. Goal: Find the equations for \( x, y, z \) or equation for the position vector \( \overrightarrow{OP} \).

Lines

Given a point on the line \( P_0(x_0, y_0, z_0) \) and a vector \( \mathbf{v} = \langle a, b, c \rangle \) that is parallel with the line, we determine that any point on the line \( P(x, y, z) \) satisfies the relation \( \overrightarrow{P_0P} = tv \). The position vector of \( P \) is given by

\[
\mathbf{r} = \overrightarrow{OP_0} + tv.
\]

This is called the vector equation of the line.

Parametric equations

\[
x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.
\]

Symmetric equations (assume \( a, b, c \) are all nonzero):

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\]

Example: (1). Find the parametric and symmetric equations of the line \( L \) through \( P_0(3, 1, -2) \) and \( P_1(4, -1, 1) \). (2). Find a parametric equation for the line segment \( P_0P_1 \).

(For (2), find the range of \( t \))

Example: Consider the skew lines \( L_1 : x = 1 + t, y = -1 - t, z = 2 - 3t \) and \( L_2 : x = t, y = t + 1, z = 2 - t \). What is the distance between them?

Exercise: Suppose we have a line given by \( \mathbf{r} = \mathbf{r}_0 + tv \). Find the distance from \( Q(x, y, z) \) to the line.

Planes

Suppose \( \mathbf{n} = \langle a, b, c \rangle \) is perpendicular with Plane \( \mathcal{P} \) (this is called normal vector) and \( P_0(x_0, y_0, z_0) \) is on the plane. Then, \( \overrightarrow{P_0P} \perp \mathbf{n} \). The vector equation:

\[
(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,
\]
where \( \mathbf{r} = \overrightarrow{OP} \) and \( \mathbf{r}_0 = \overrightarrow{OP}_0 \).

**Scalar equation:**
\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
\]

**Angle between two planes**

If two planes have normal vectors \( \mathbf{m} \) and \( \mathbf{n} \) respectively, the angle between them satisfies
\[
\cos \theta = \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}||\mathbf{n}|}.
\]

**Exercise:** Draw a picture to illustrate that this formula is true.

**Example:** Suppose \( A(1, 0, -1), B(3, 3, 2), C(4, 5, -1) \) and \( D(0, 0, 1) \).

- Find the scalar equation for the plane \( ABC \)
- Find the distance from point \( D \) to the plane \( ABC \)

**Solution.** (1). A normal vector is \( \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -15, 9, 1 \rangle \). If we use the point \( A(1, 0, -1) \), then the equation will be 
\[
-15(x - 1) + 9(y - 0) + 1(z + 1) = 0 \text{ or } -15x + 9y + z + 16 = 0.
\]

(2). \( d = |\overrightarrow{AD} \cdot \mathbf{n}|/|\mathbf{n}|. \)

**Example:** The two lines \( L_1 : \frac{x - 1}{2} = \frac{y - 3}{3} = z - 1 \) and \( L_2 : \frac{x - 1}{3} = \frac{y - 3}{2} = z - 1 \) intersect at a point.

- Find the scalar equation of the plane that contains these two lines.
- Find a plane that passes through \( A(-1, 1, 2) \) and \( B(-2, 3, 3) \) and is parallel with \( L_1 \).

**Solution.** Let \( \mathbf{v}_1 = \langle 2, 3, 1 \rangle \) and \( \mathbf{v}_2 = \langle -1, 2, 1 \rangle \). A normal vector is \( \mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2. \)

**12-5: Curves and motion in space**

A point moves in space and it traces out a curve. The coordinates of that point can be written as
\[
x = x(t), \quad y = y(t), \quad z = z(t).
\]

In other words, the position vector can be written as
\[
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}.
\]

Hence, a curve is described by a vector-valued function.
Continuity, Differentiation, Integration

We can simply look at each component since \( i, j, k \) are independent of time \( t \).

For example, the derivative is defined to be

\[
 r'(t) = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} = \langle x'(t), y'(t), z'(t) \rangle.
\]

Proof.

\[
 r'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ (x(t+\Delta t)i + y(t+\Delta t)j + z(t+\Delta t)k) - (x(t)i + y(t)j + z(t)k) \right]
 = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} i + \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t} j + \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t} k
 = x'(t)i + y'(t)j + z'(t)k.
\]

Another example: the integration \( \int_a^b r(t)dt \) is defined to be

\[
 \int_a^b r(t)dt = \lim_{\Delta t \to 0} \sum_{i=1}^n r(t^*_i)\Delta t = \left( \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right),
\]

where we divide \([a, b]\) into \( n \) subintervals and \( t^*_i \) is a sample point from the \( i \)-th subinterval.

Exercise: Read P809 and understand why we can integrate each component to get the integral. If \( r = u(t)\hat{a}(t) + v(t)\hat{b}(t) \), is it true or false that \( \int_0^1 r(t)dt = \hat{a}(t) \int_0^1 u(t)dt + \hat{b}(t) \int_0^1 v(t)dt \)?

(The answer is false. Here, the two unit vectors are changing.)

Velocity and acceleration

If a particle is moving with position vector \( r(t) \), then

\[
 v(t) = r'(t)
 a(t) = v'(t) = r''(t).
\]

They are called the velocity and acceleration.

The speed: \( v(t) = |v(t)| \). The scalar acceleration \( a(t) = |a(t)| \).
Differentiation formulas

- \((h(t)u(t))' = h'(t)u(t) + h(t)u'(t)\)
- \((u(t) \cdot v(t))' = u'(t) \cdot v(t) + u(t) \cdot v'(t)\)
- \((u(t) \times v(t))' = u'(t) \times v(t) + u(t) \times v'(t)\)

Let's show the second for vectors with 3 components.

Proof.

\[
(u(t) \cdot v(t))' = \left(\sum_{i=1}^{3} u_i(t)v_i(t)\right)' = \sum_{i=1}^{3} (u_i'(t)v_i(t))
\]

\[
= \sum_{i=1}^{3} (u_i'(t)v_i(t) + u_i(t)v_i'(t)) = \left(\sum_{i=1}^{3} u_i'(t)v_i(t)\right) + \left(\sum_{i=1}^{3} u_i(t)v_i'(t)\right)
\]

\[
= u'(t) \cdot v(t) + u(t) \cdot v'(t).
\]

Exercise: Explain in words why each step in the proof holds.

Two claims:

- If \(u'(t) \perp u(t)\), then \(|u(t)|\) is a constant.
- \(|u(t)'| = u(t) \cdot u'(t)/|u(t)|\).

Example: A particle is moving on the unit sphere. Show that its velocity is perpendicular with the position vector.

Example: A charged particle in a magnetic field feels the Lorentz force \(F = (qv) \times B\). Show that the charged particle in a magnetic field does not change the speed.

Solution.

\[
F \cdot v = 0 \Rightarrow a \cdot v = 0 \Rightarrow \frac{d}{dt} |v|^2 = 0
\]

Example: A moving particle has a given initial position \(r(0) = 2i\) with initial velocity \(v(0) = i - j\). Suppose the acceleration is \(a(t) = 2i + 6tj\). Find the velocity and position at time \(t\)