15.7 Stokes’ Theorem

This is the generalization of the first version of Green’s theorem to closed curves in 3D space.

**Theorem 1.** Let $S$ be an oriented surface with unit normal vector $n$. $C$ is its boundary and thus a closed curve in 3D space. If we look into the surface from the direction where $n$ points to (this means $n$ is pointing into your eyes), $C$ is counterclockwise. Let $F$ be a continuously differentiable vector field, then

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, dS = \iint_S \text{curl}(F) \cdot n \, dS.$$

In coordinates form, we have

$$\oint_C Pdx + Qdy + Rdz = \iint_S \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \cdot n \, dS$$

$$= \iint_S (R_y - Q_z)dydz + (P_z - R_x)dzdx + (Q_x - P_y)dxdy$$

The proof is to use the parametrization and apply Green’s theorem for the double integral over $u, v$. We’ll not do the proof here.

*Exercise:* Verify that when $S$ is the surface in $xy$ plane, this agrees with Green’s theorem.

- The integral on the left is called the circulation of $F$.
- $C$ is the boundary of $S$, or $C = \partial S$.
- The orientations of $C$ and $S$ must agree. If the normal of $S$, $n$, is pointing into your eyes, the curve $C$ must be counterclockwise. Otherwise, we have a negative sign.
- The surface we use doesn’t matter. We get the same answer as long as it has the same boundary. This is the so-called surface independence.

*Example:* Let $C$ be the intersection of $z = y + 3$ and $x^2 + y^2 = 1$, oriented counterclockwise of viewed from above. Let

$$F = 3zi + 5xj - 2yk.$$ 

Compute $\oint_C F \cdot T \, ds$ in two ways.
Physical meaning of curl

We can divide the surface $S$ into $N$ pieces. One piece is $S_i$, with boundary $C_i = \partial S_i$. Then, we find

$$\oint F \cdot dr = \sum_{i=1}^{N} \oint_{C_i} F \cdot dr = \sum_{i=1}^{N} \Delta S \left( \frac{1}{\Delta S} \int_{S_i} (\nabla \times F) \cdot n dS \right)$$

The first equality holds because the line integrals on the inner edges cancel out exactly. As $\Delta S \to 0$.

$$(\nabla \times F) \cdot n = \lim_{|S| \to 0} \frac{1}{\Delta S} \oint_{\partial S_i} F \cdot dr.$$ 

curl($F$) is the directed density of circulation and it indicates the direction and rate of rotation of the field near the point. $Q_x - P_y$ in Green’s theorem is just 2D curl and it has the same meaning.

Irrotational fields

If $\nabla \times F = 0$, then, it’s called irrotational.

Previously, we find that if $F$ is conservative, then $F = \nabla \phi$ it’s irrotational by Clairaut’s theorem. The reverse is true.

We show the reverse direction. We show that the line integral of $F$ is path independent. Let $C_1, C_2$ be two curves with the same endpoints. Then, $C = C_1 - C_2$ is a closed curve.

$$\int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = \oint_C F \cdot dr = \int_S (\nabla \times F) \cdot n dS = 0.$$ 

To summarize

**Theorem 2.** Suppose $D$ is simply connected (no holes). $F$ is irrotational if and only if it is conservative or $F = \nabla \phi$.

**Example:** Let $F = \langle 3x^2, 5z^2, 10yz \rangle$. Is it conservative? If yes, find a potential and evaluate

$$\int_C F \cdot T ds$$

where $C$ is given by $r = \langle \ln(1 + t^3), 2t^8 + 1, \sin(2t) \rangle, 0 \leq t \leq 1$

The field is irrotational. Hence, you can find a potential. Then, use the fundamental theorem.
More examples

Example: Compute \( \iint_S (\nabla \times F) \cdot n dS \), where \( F = \langle 3z, 5x, -2y \rangle \) and \( S \) is \( z = x^2 + y^2 \) under \( z = 4 \). The normal is the one pointing up.

You can compute the curl, parametrize the surface, and compute the flux as a surface integral. Here, we can simply reduce the flux to the circulation by Stokes’s theorem. The curve is a circle \( r(t) = (2 \cos t, 2 \sin t, 4) \)

Example: Let \( F = \nabla \times \langle yz, -xz, z \rangle \) and \( S \) is \( x^2 + y^2 = e^z \) between \( z = 0 \) and \( z = 2 \ln 2 \), with the normal point upward. Compute

\[
\iint_S F \cdot n dS
\]

Example: Suppose \( \vec{G} = \langle e^{x^2} + 3y + z, \sin(y^3) + x, 4y - 3z \rangle \). \( C \) is the intersection of \( z = x^2 + y^2 \) and \( z = 2x + 4y \) oriented \textbf{clockwise} when viewed from above. Compute the circulation of \( \vec{G} \) over \( C \).

The straightforward way is to change it into the flux of curl by Stokes’s. Here, you can either use the surface from the paraboloid or the plane. Obviously, the plane is the one we desire. A little bit thing: before you compute, you can throw away the conservative filed \( \langle e^{x^2}, \sin(y^3), -3z \rangle \) to make your life easier. This is because the circulation of a conservative field is zero.